Subgraph Frequencies:
The Empirical and Extremal Geography of Large Graph Collections

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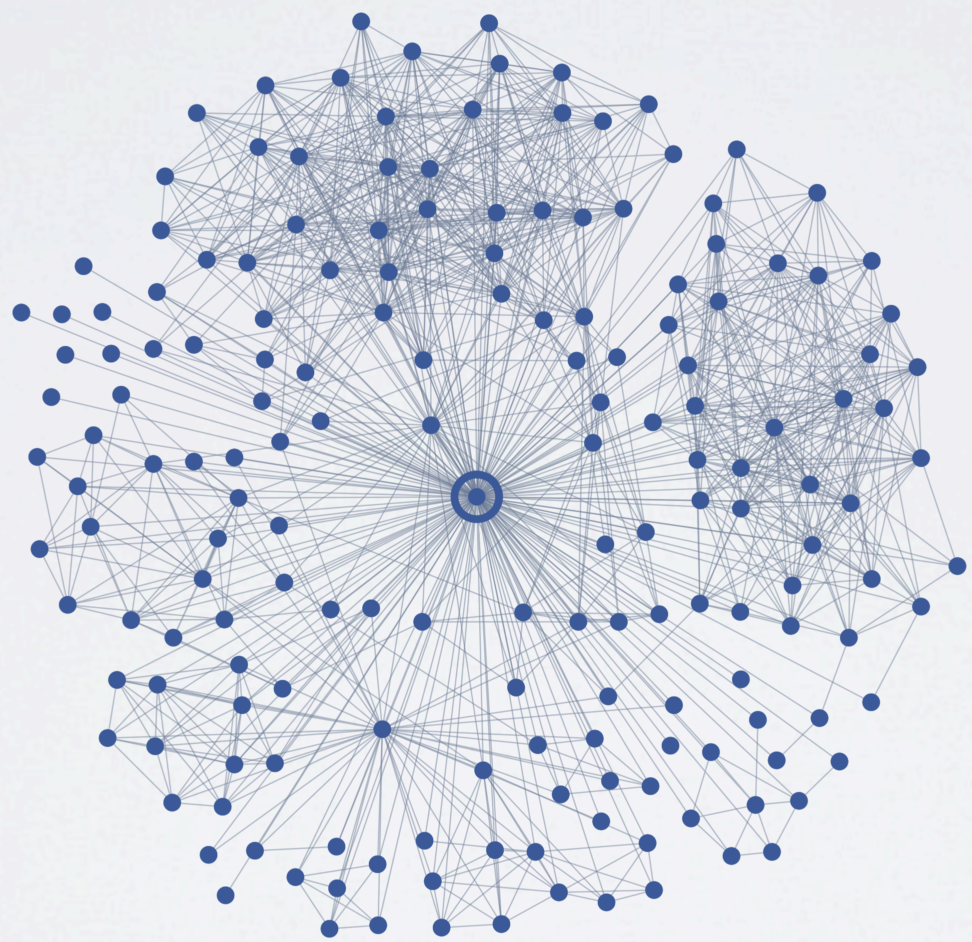
Cornell University

Graph collections

- **Neighborhoods:** graph induced by friends of a single ego, excluding ego



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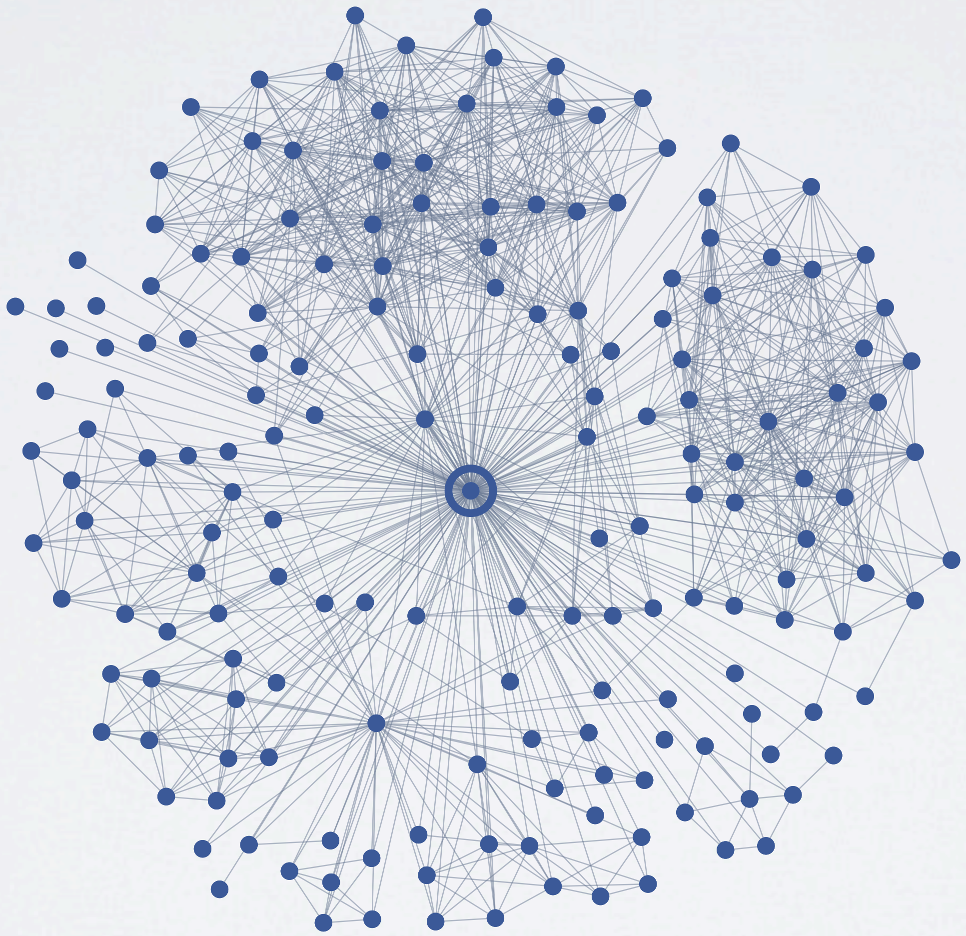
- **Groups:** graph induced by members of a Facebook ‘group‘
- **Events:** graph induced by ‘Yes’ respondents to a Facebook ‘event’



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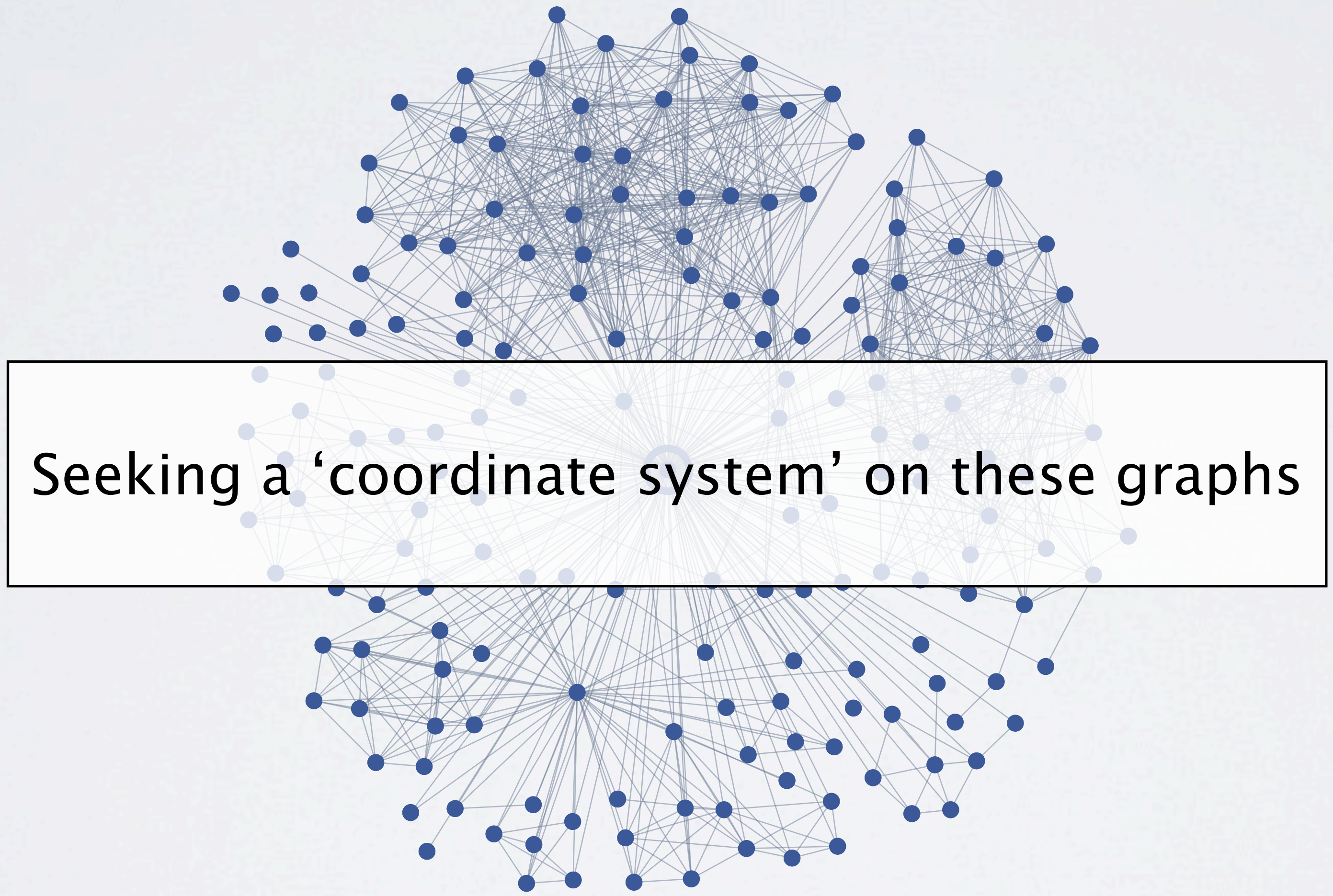
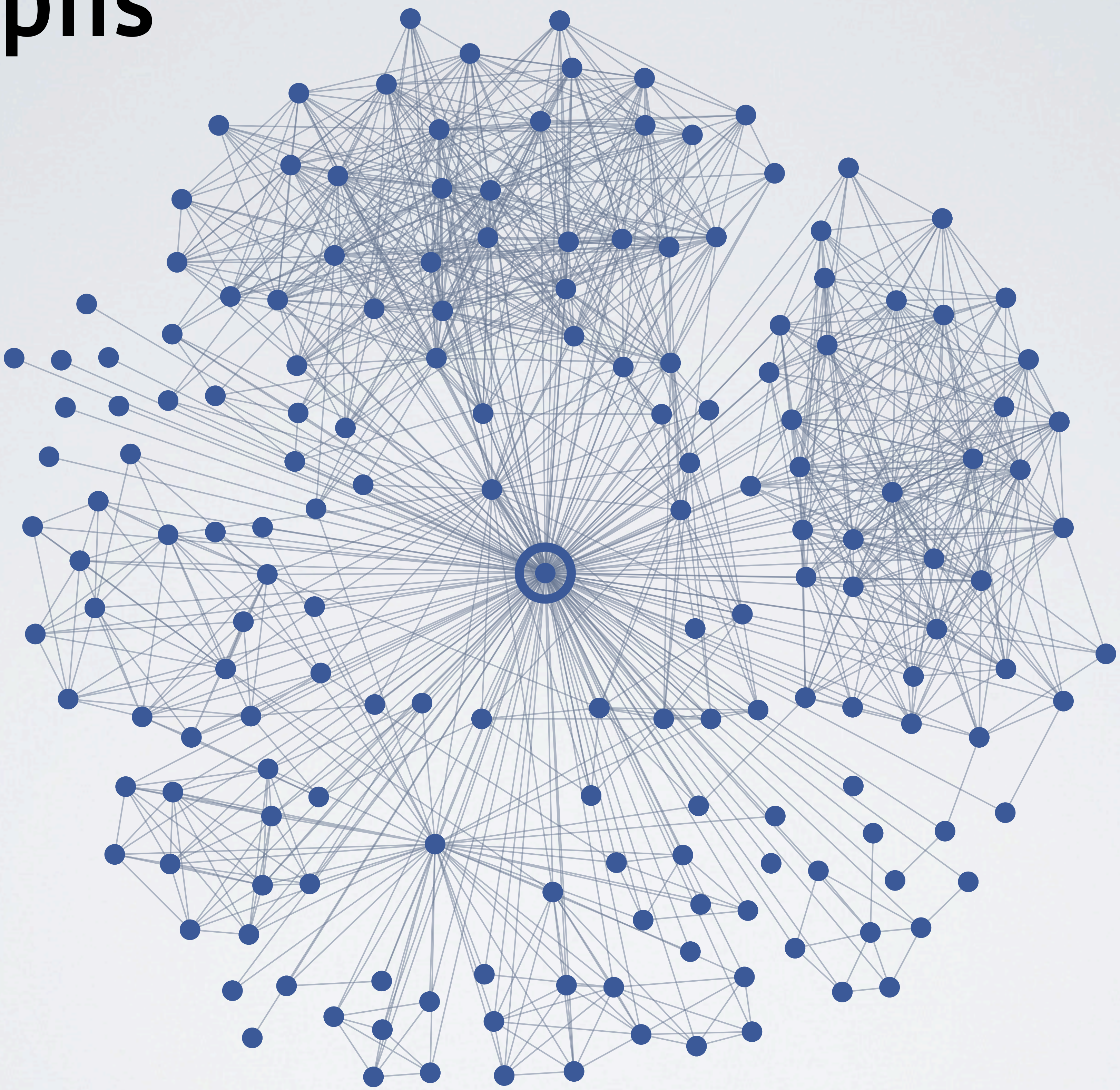


Subgraphs



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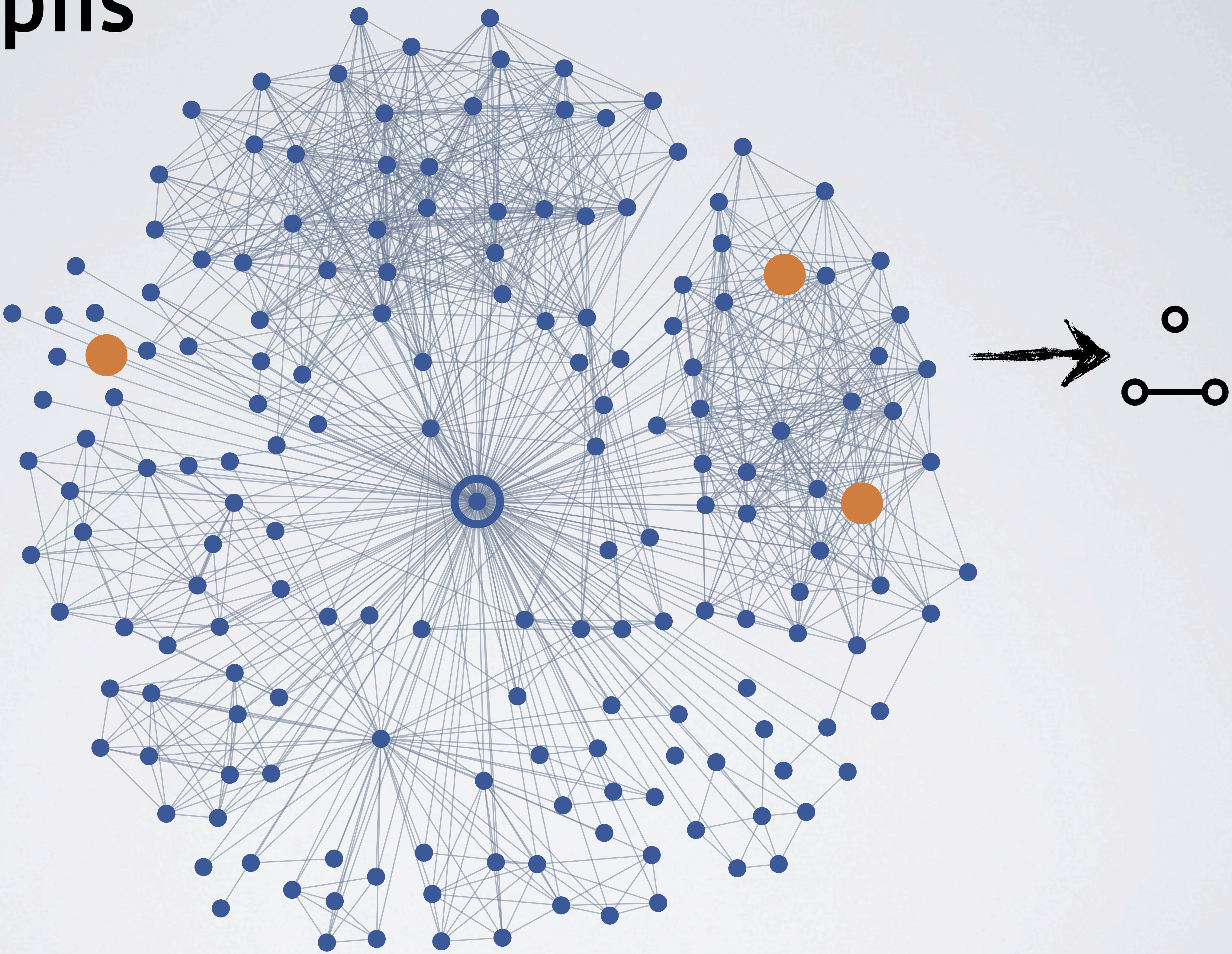

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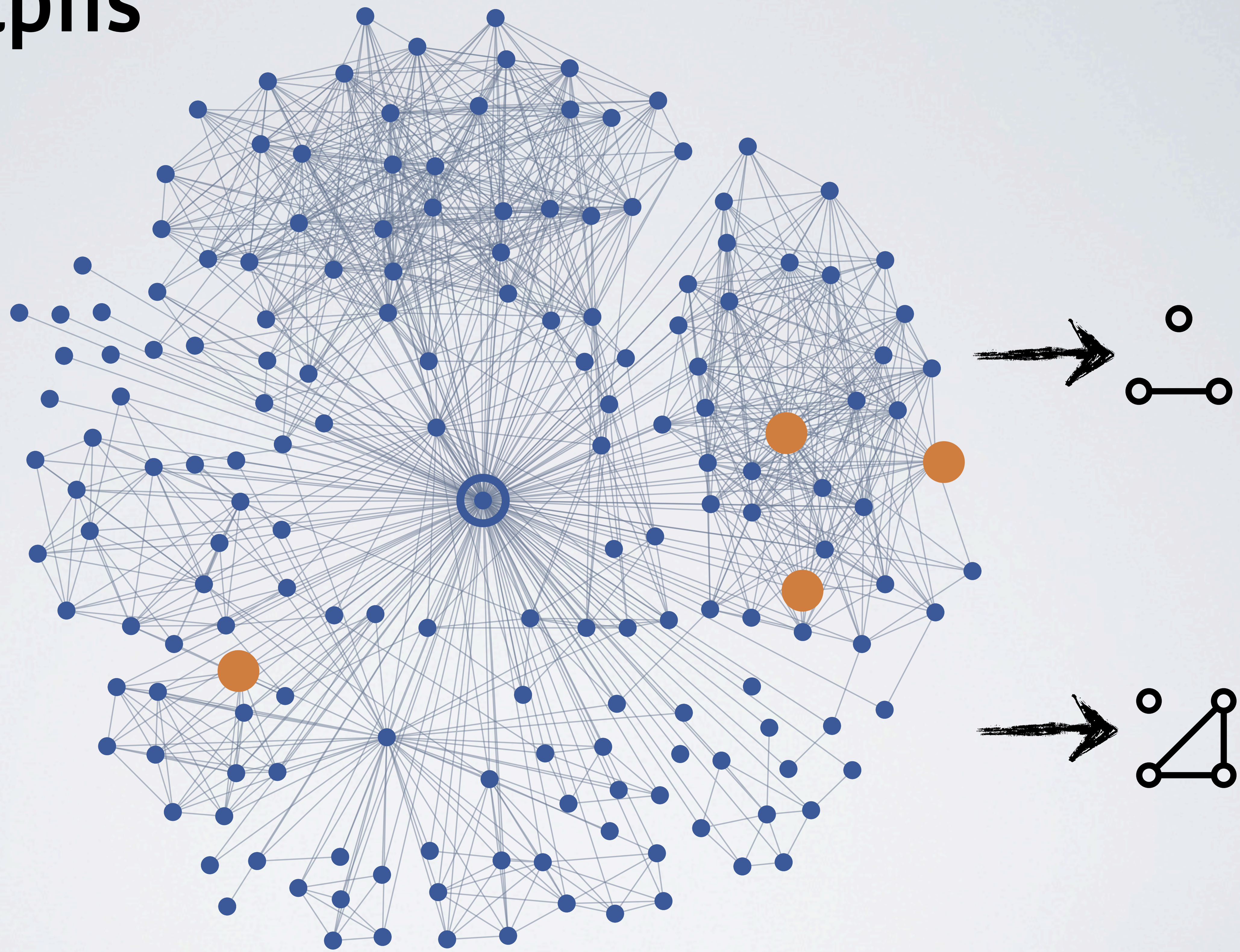

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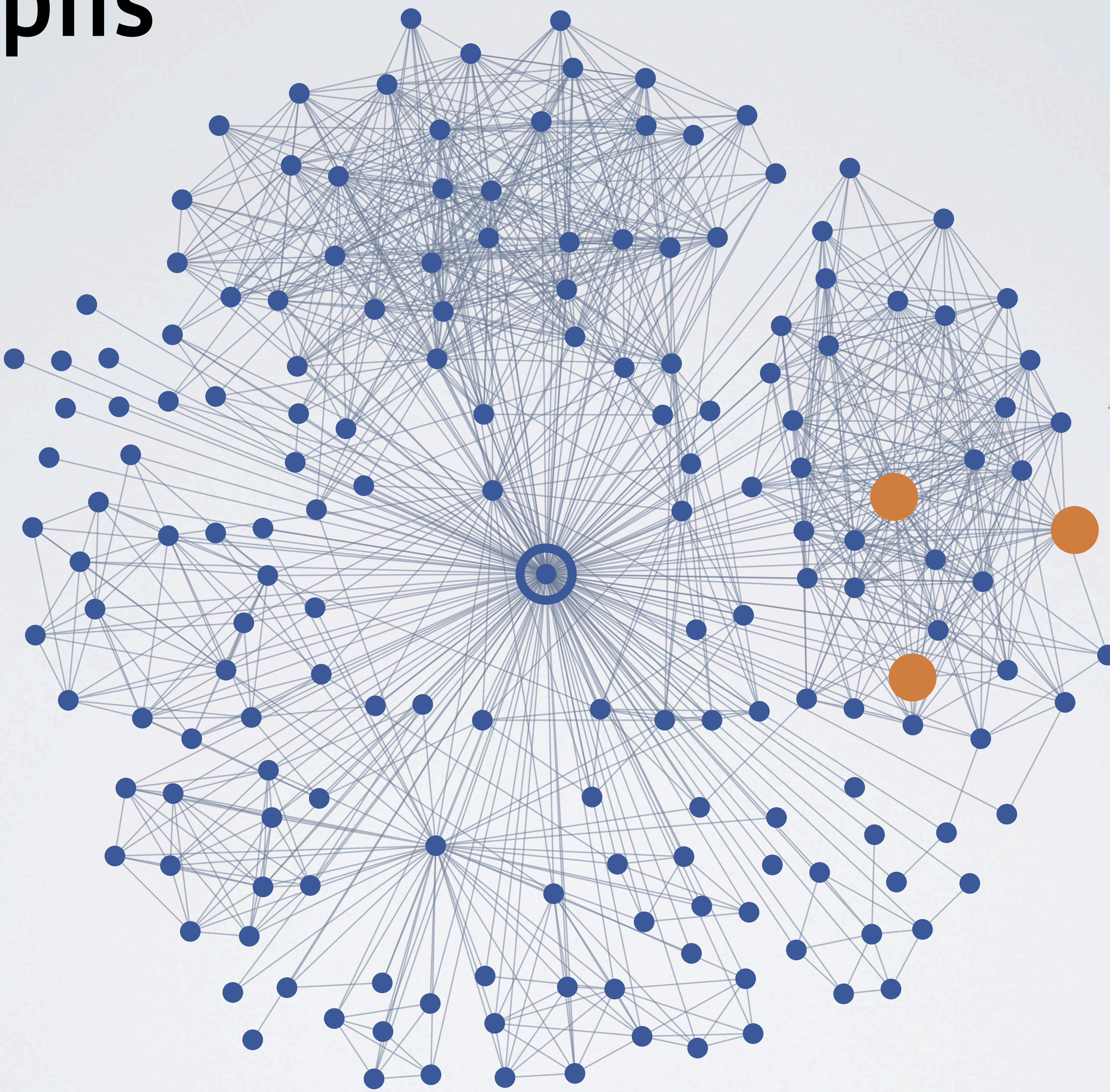
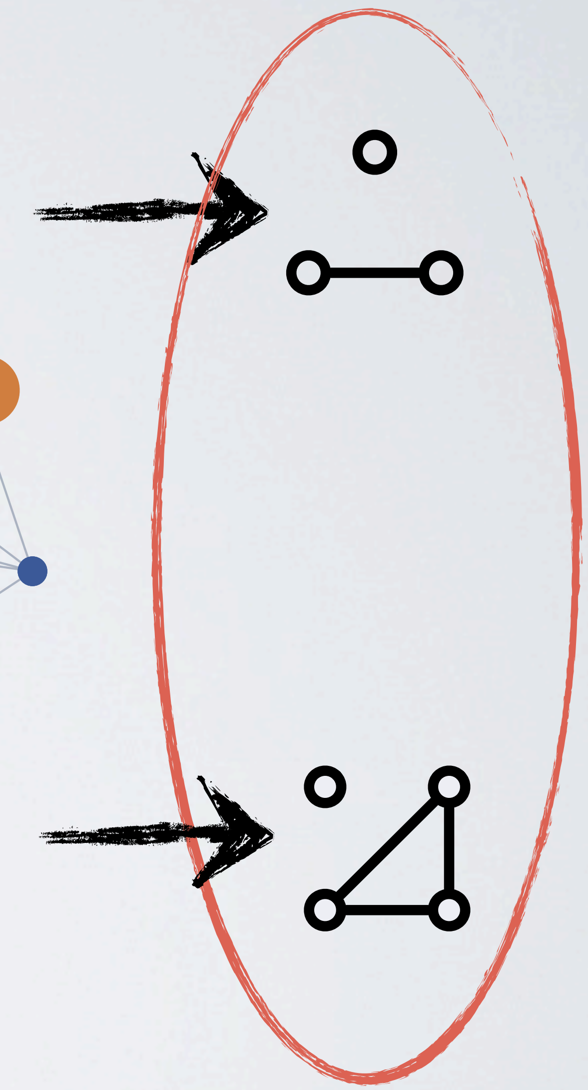


Compute frequencies



Subgraph Frequencies

- **Definition:** The **subgraph frequency s(F,G)** of a k-node subgraph **F** in a graph **G** is the fraction of k-tuples of nodes in **G** that induce a copy of **F.**

Triad census: Davis-Leinhardt 1971, Wasserman-Faust 1994
Motifs/Frequent subgraphs: Inokuchi et al. 2000, Milo et al. 2002, Yan-Han 2002, Kuramochi-Karypis 2004

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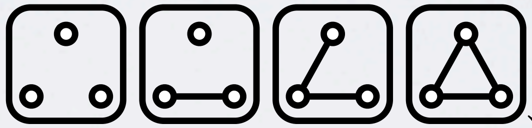

$$s(\cdot, G) = (x_1,\ x_2,\ x_3,\ x_4) = (0.18, 0.37, 0.14, 0.31)$$

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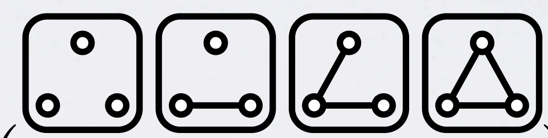

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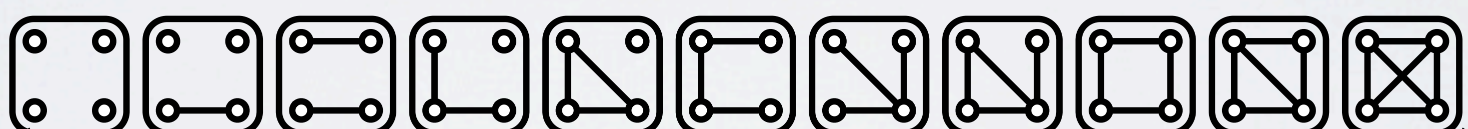

$$s(\cdot, G) = (y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6,\ y_7,\ y_8,\ y_9,\ y_{10},\ y_{11})$$

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Empirical/Extremal Questions

- Consider the subgraph frequencies as a ‘**coordinate system**’

- **Empirical Geography:**
  - What subgraph frequencies do **social graphs** exhibit?
  - Is there a good model?

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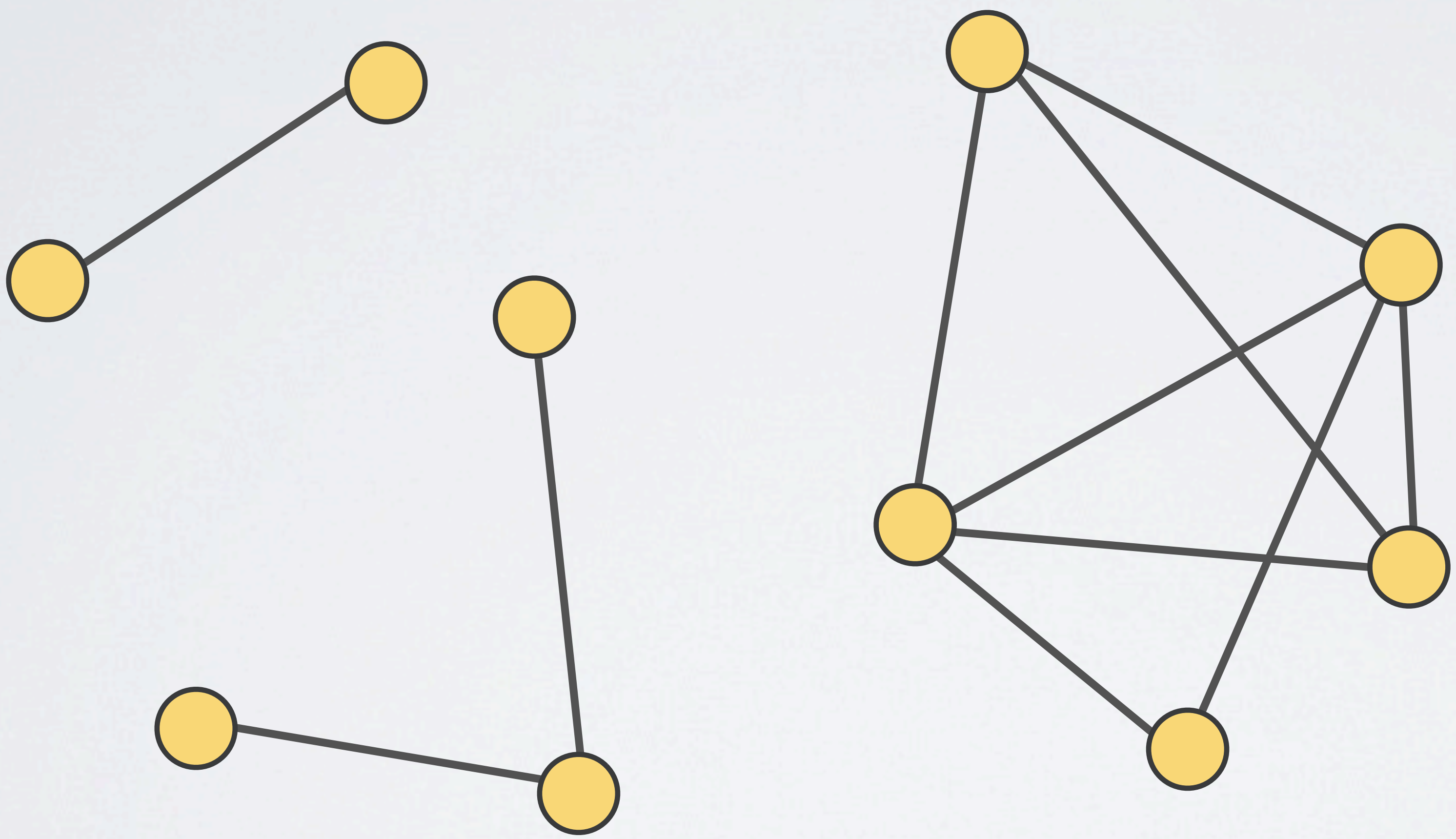
      - What's a property of graphs and what's a property of people?

What do we expect?



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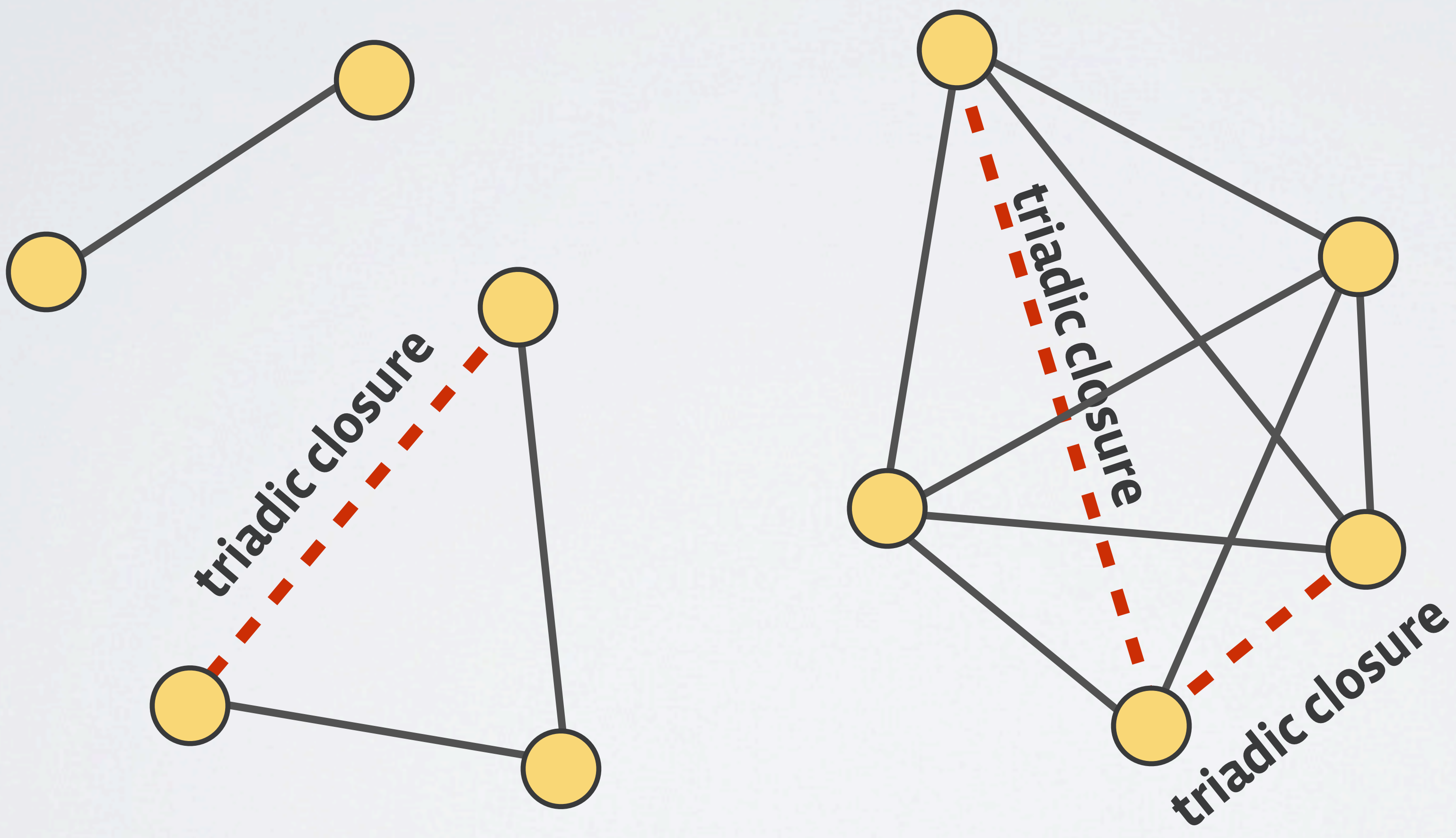
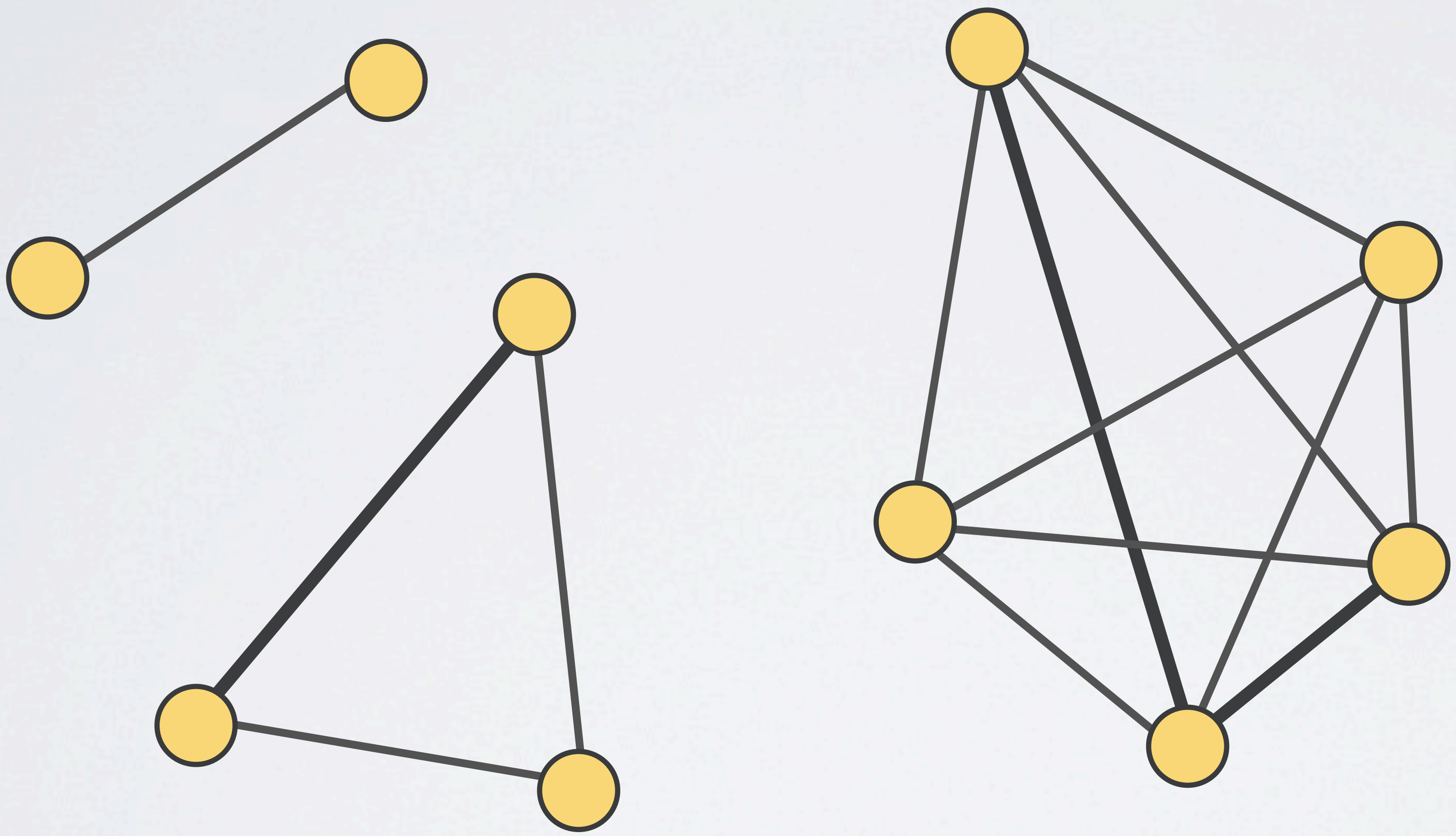
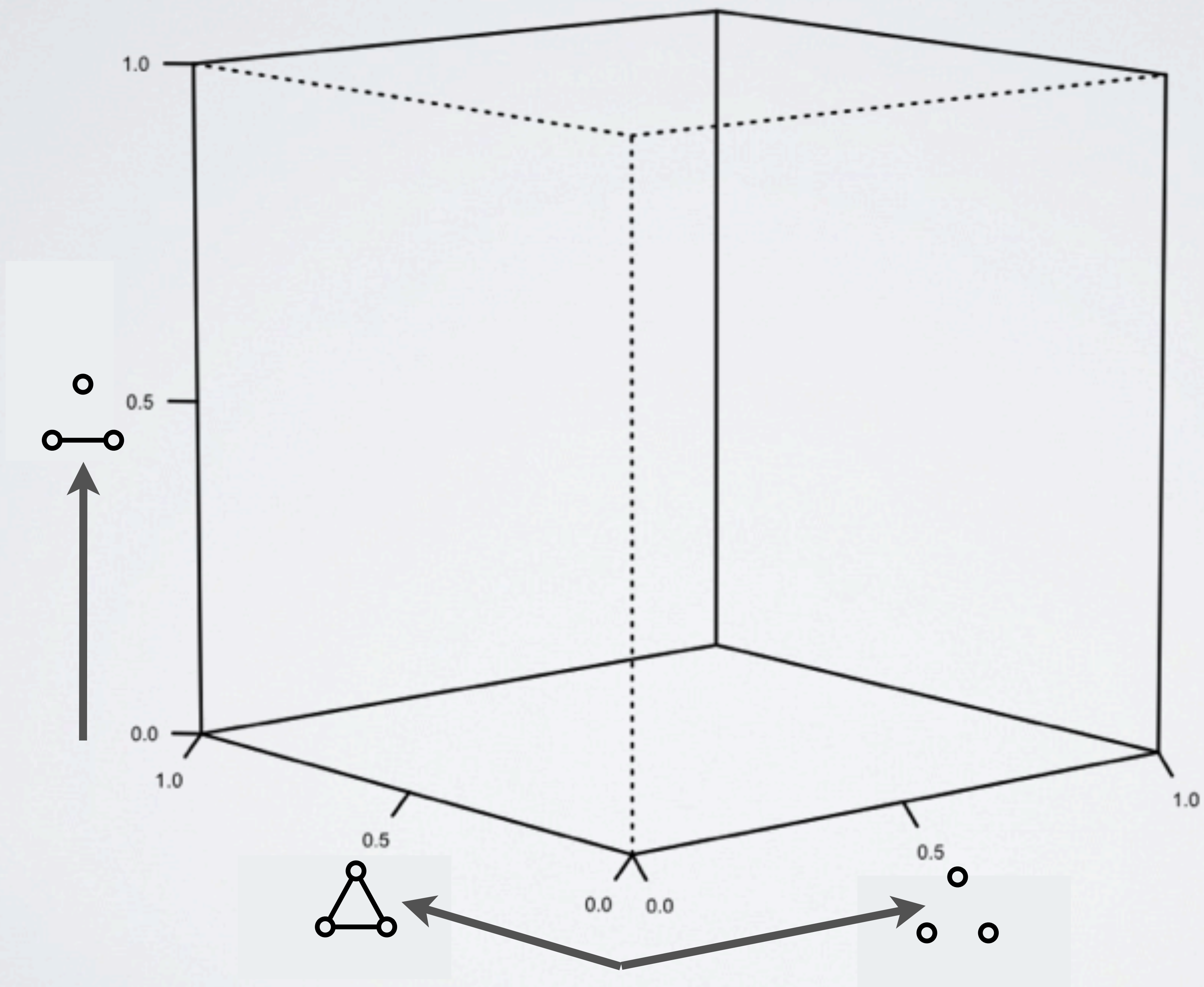


What do we expect?

We expect **few wedges, many triangles** for social networks.



The triad space



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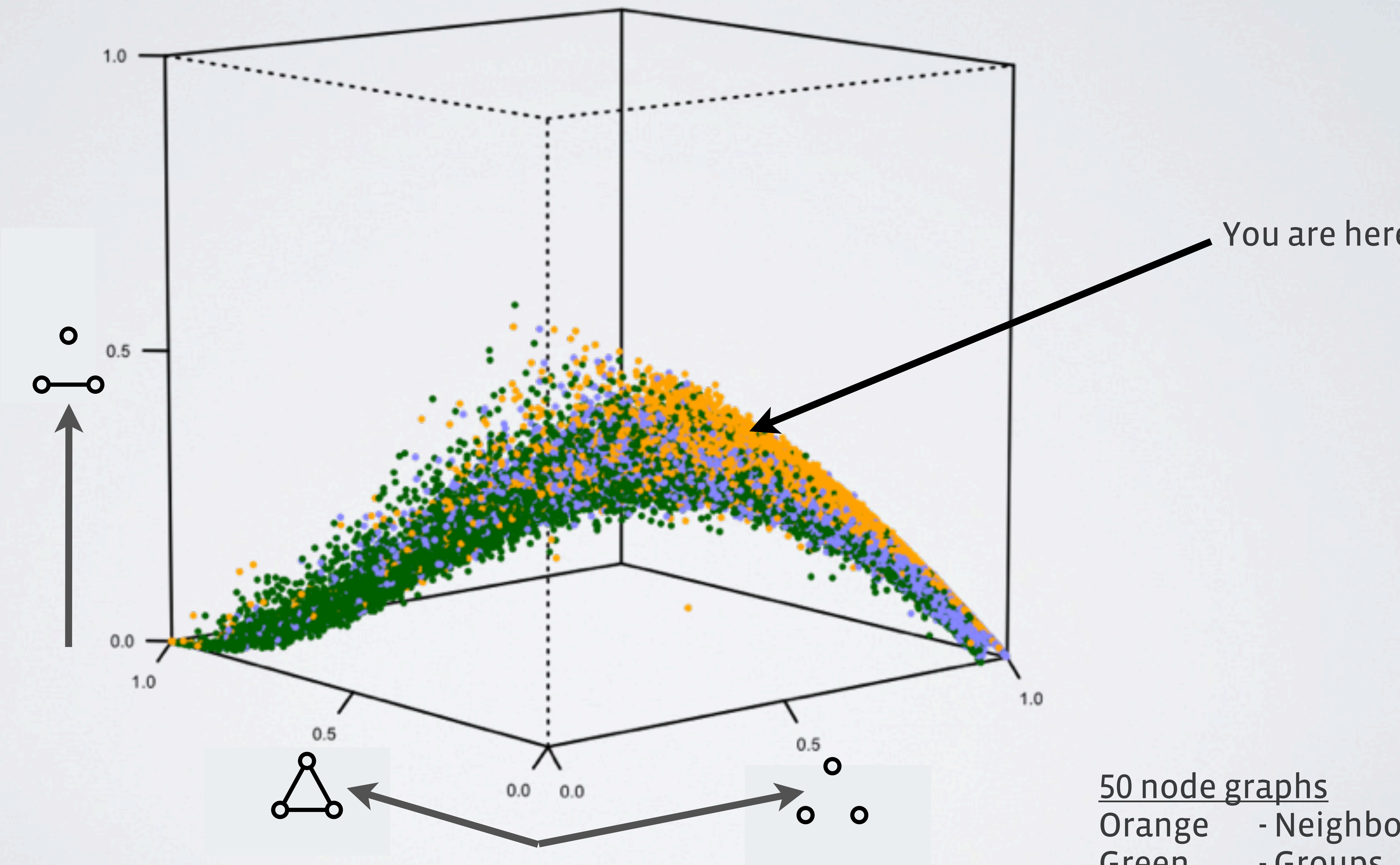

<u>50 node graphs</u>
Orange - Neighborhoods
Green - Groups
Lavender - Events

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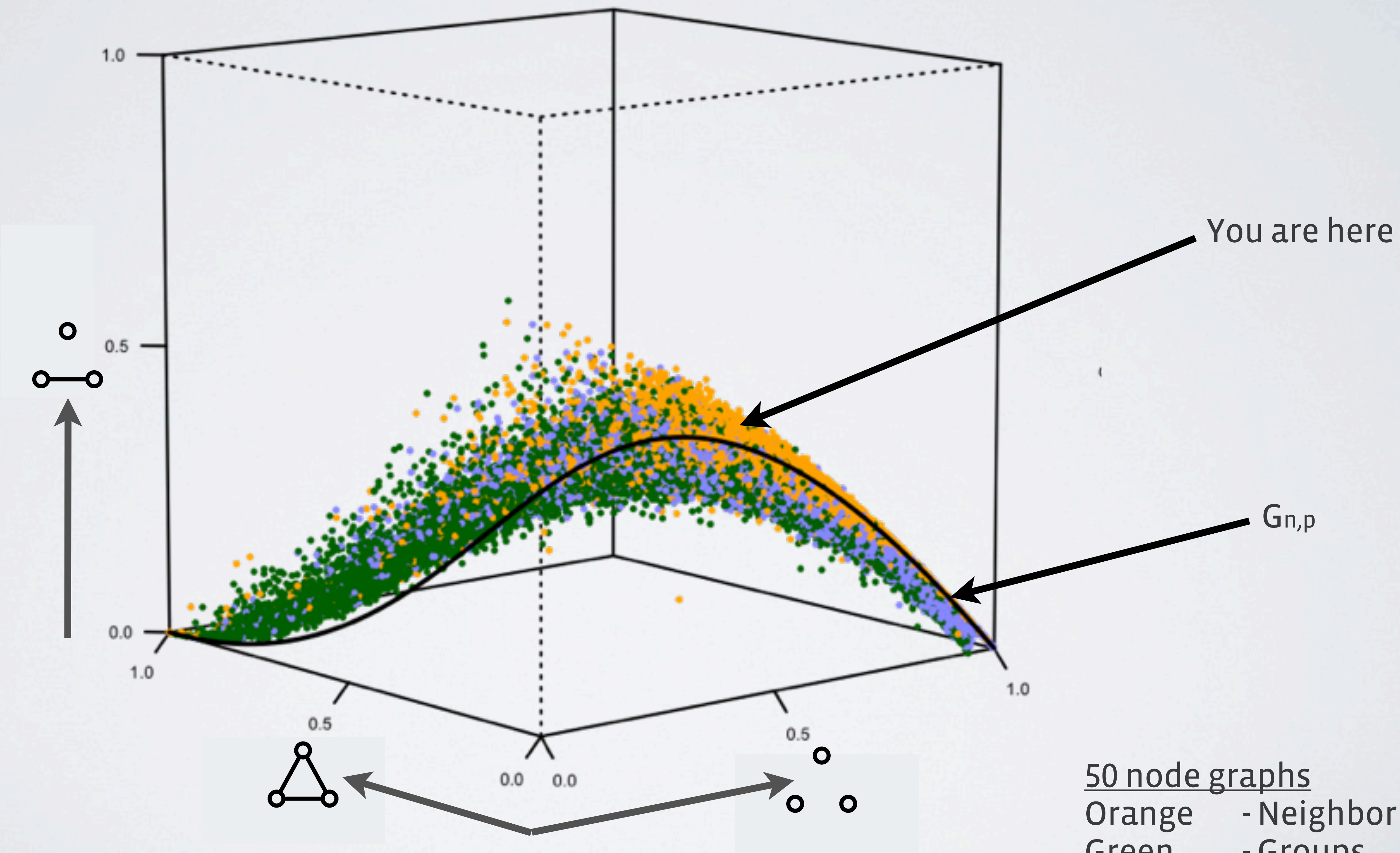

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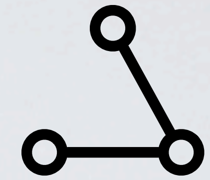
Subgraph frequency of 

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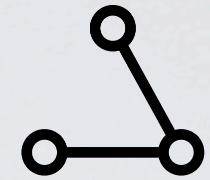
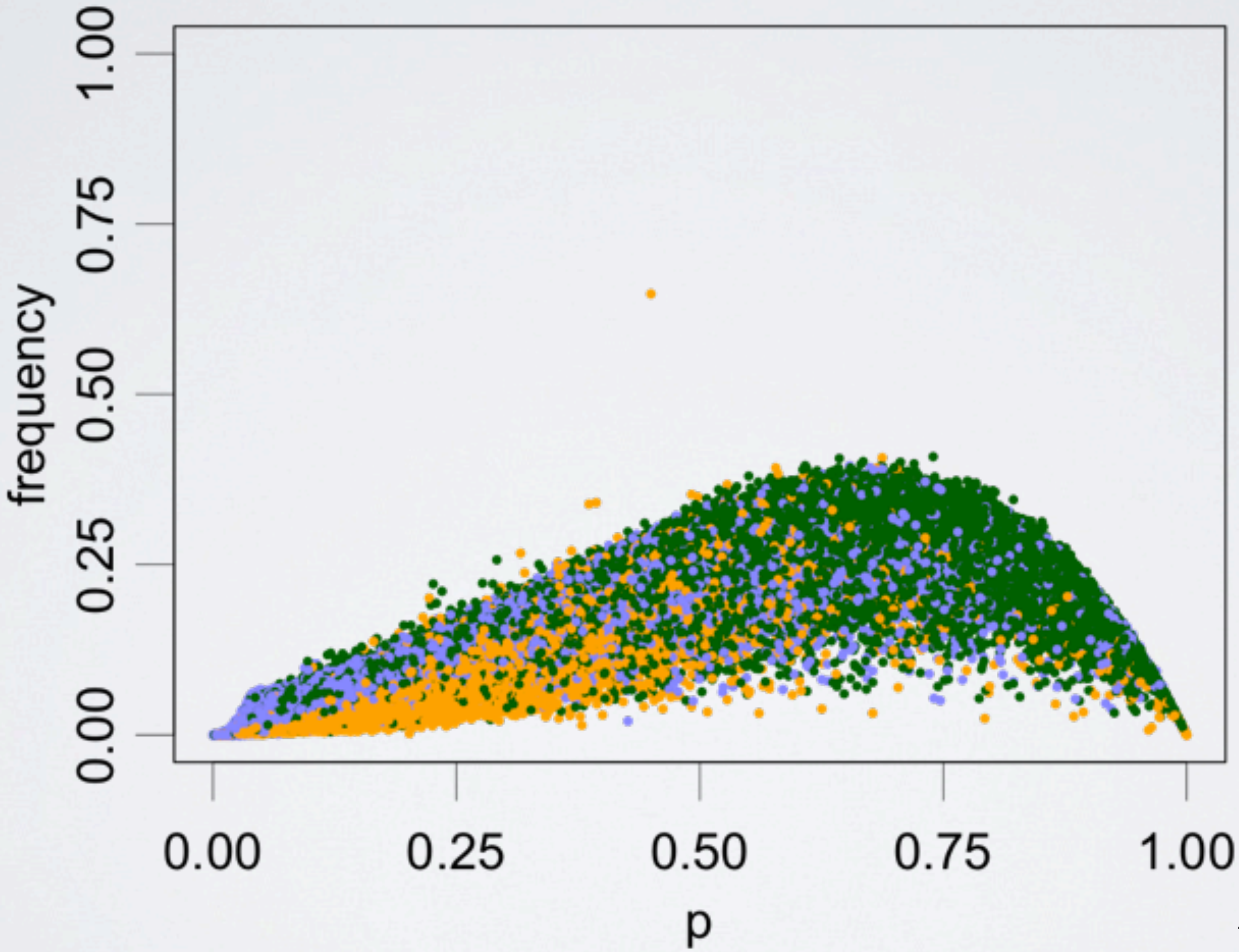
Subgraph frequency of 



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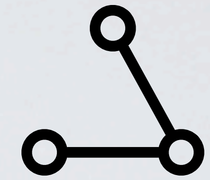
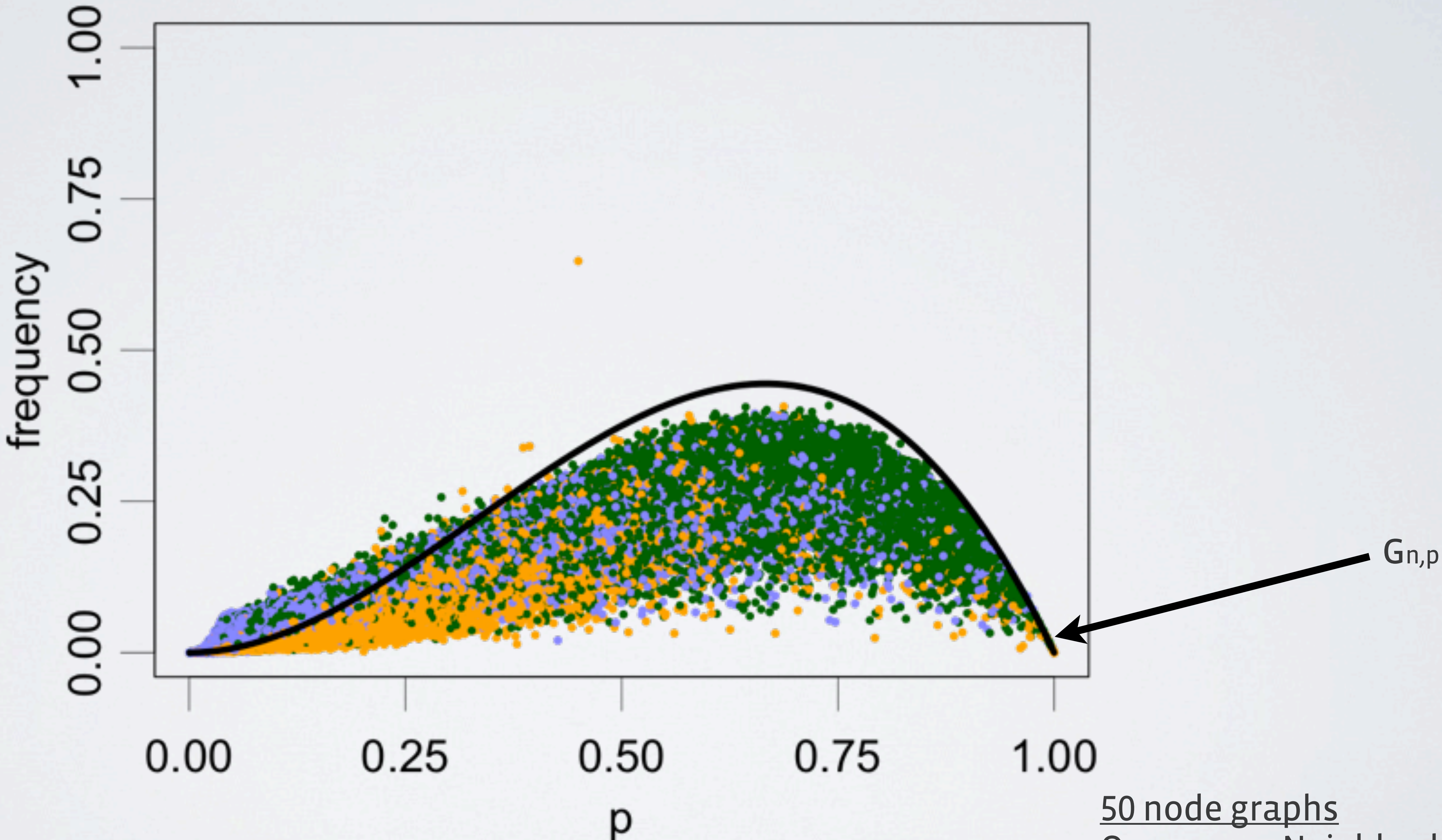
Subgraph frequency of 



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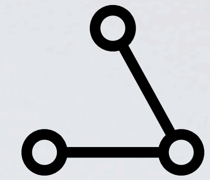
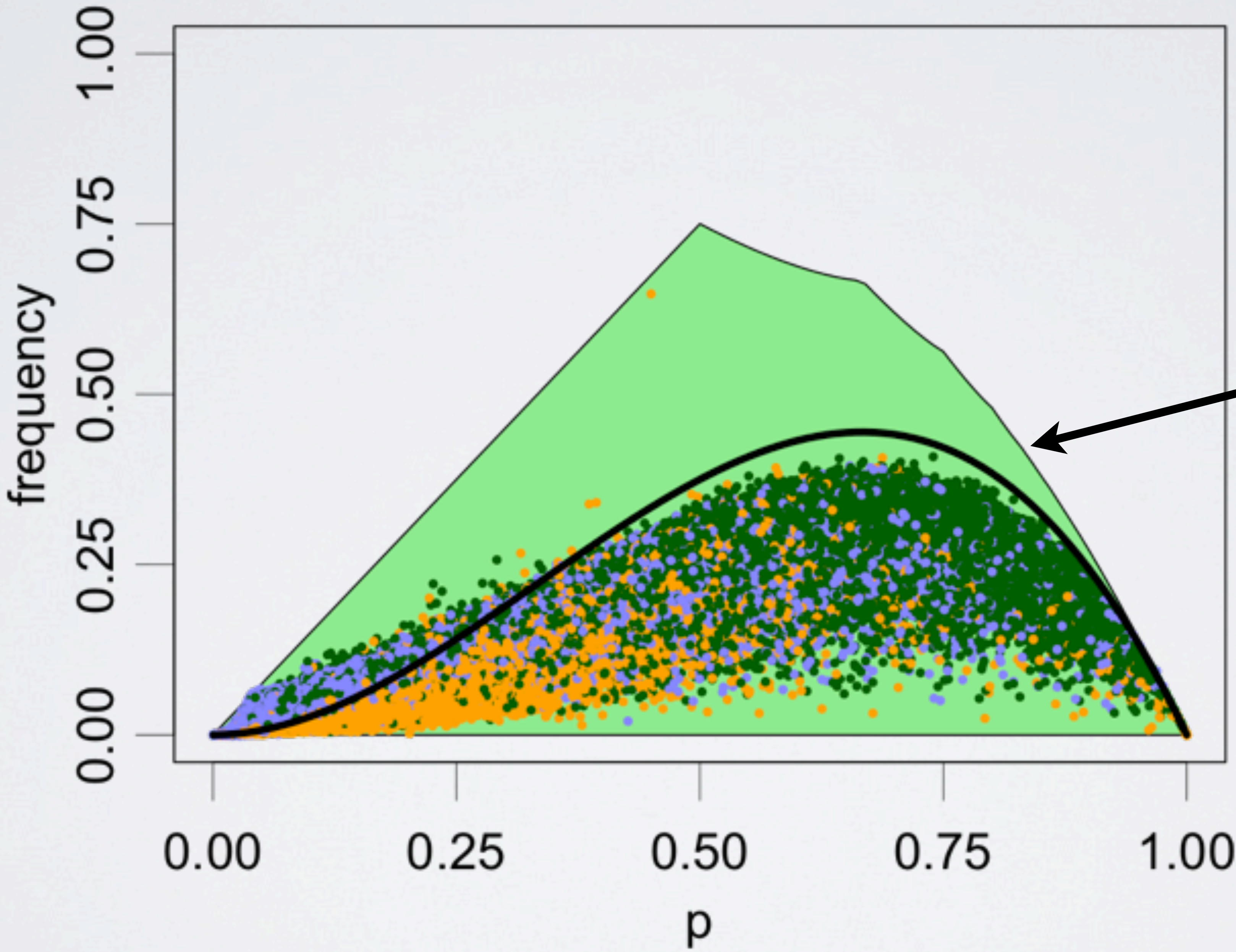
Subgraph frequency of 



Extremal
Graph Theory

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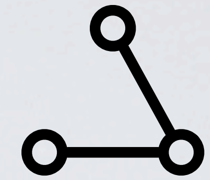
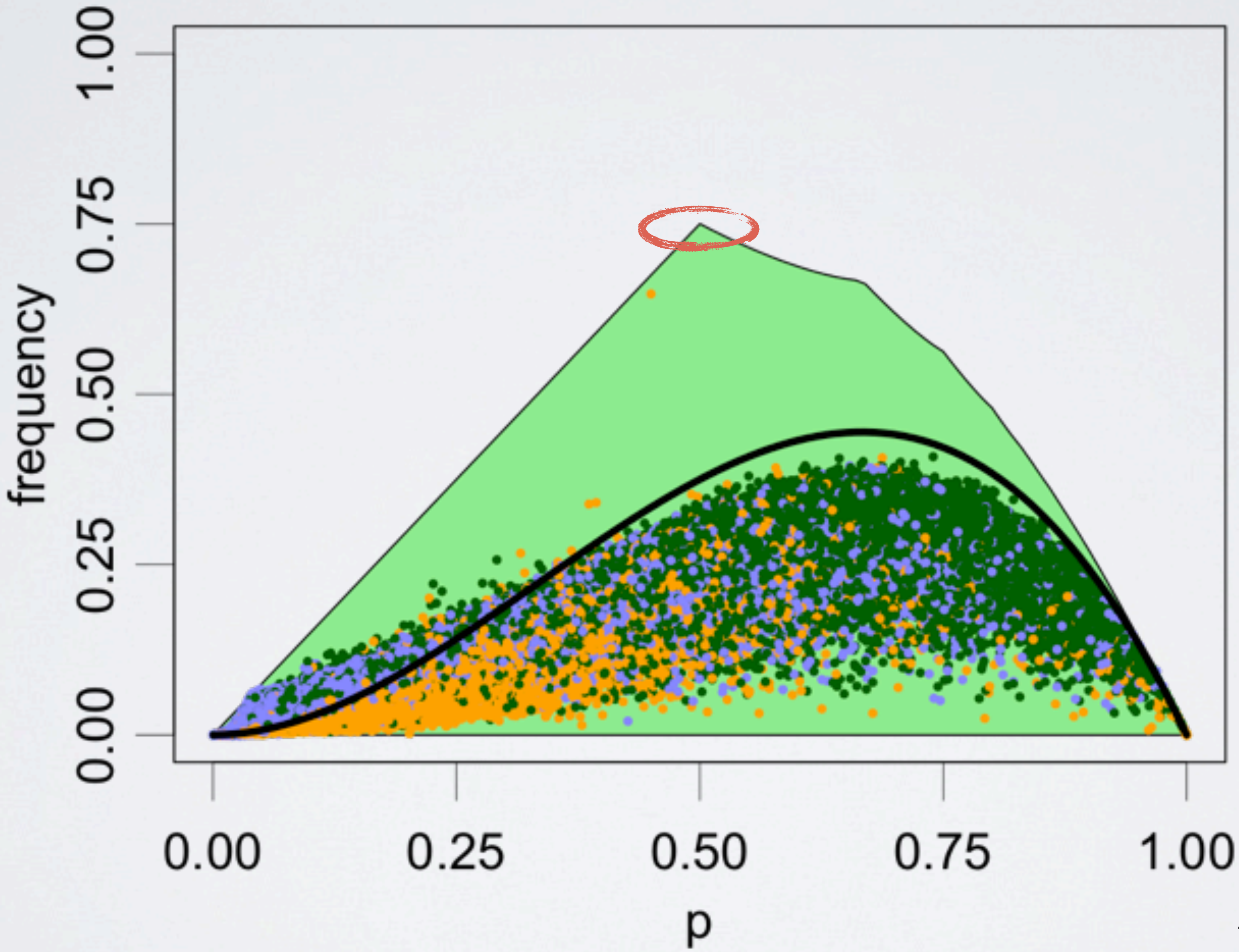
Subgraph frequency of 

Frequency of the 'forbidden triad' is bounded at ≤ **3/4**.
Sharp for $K_{n/2,n/2}$ (bipartite graph) when n is even.



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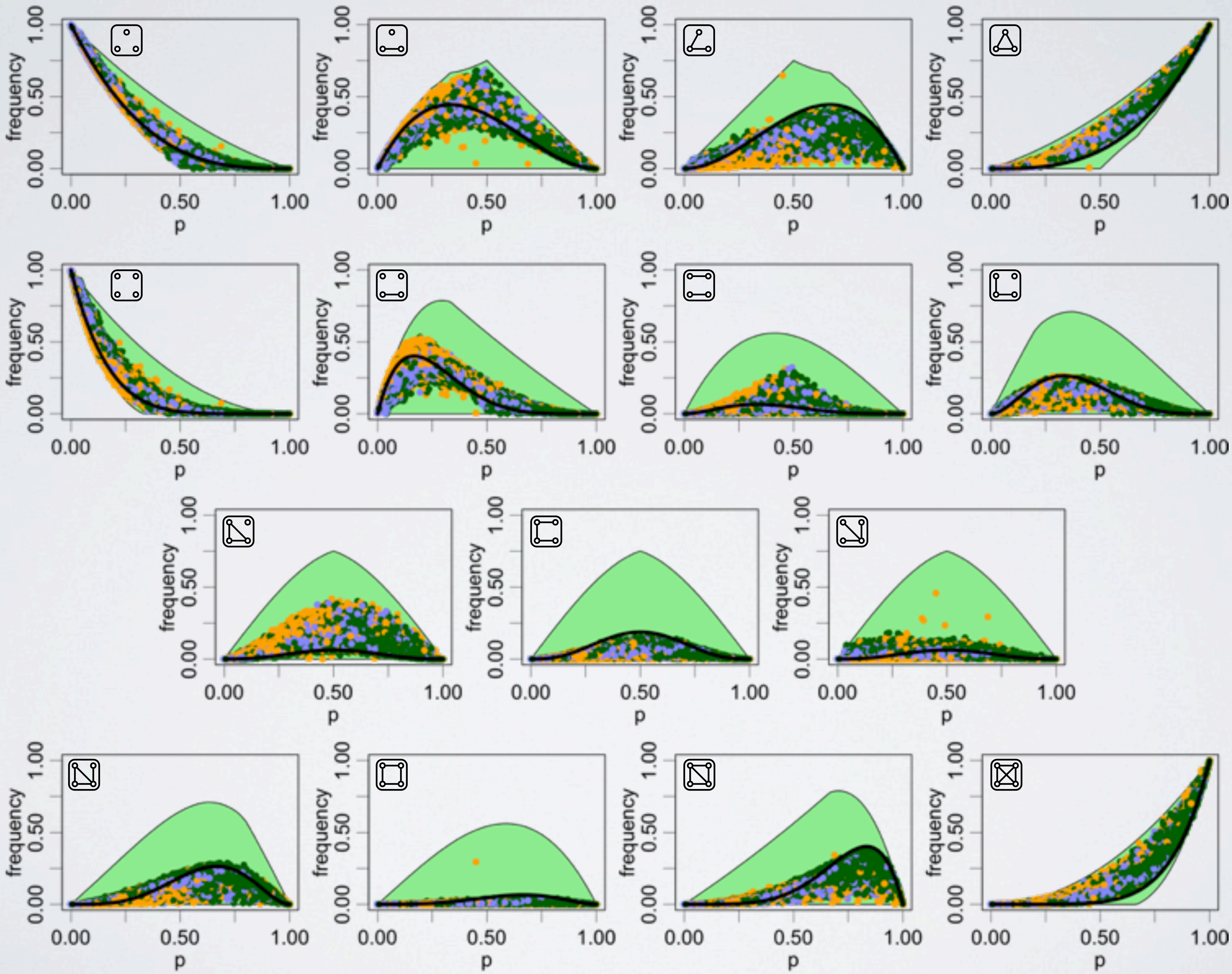
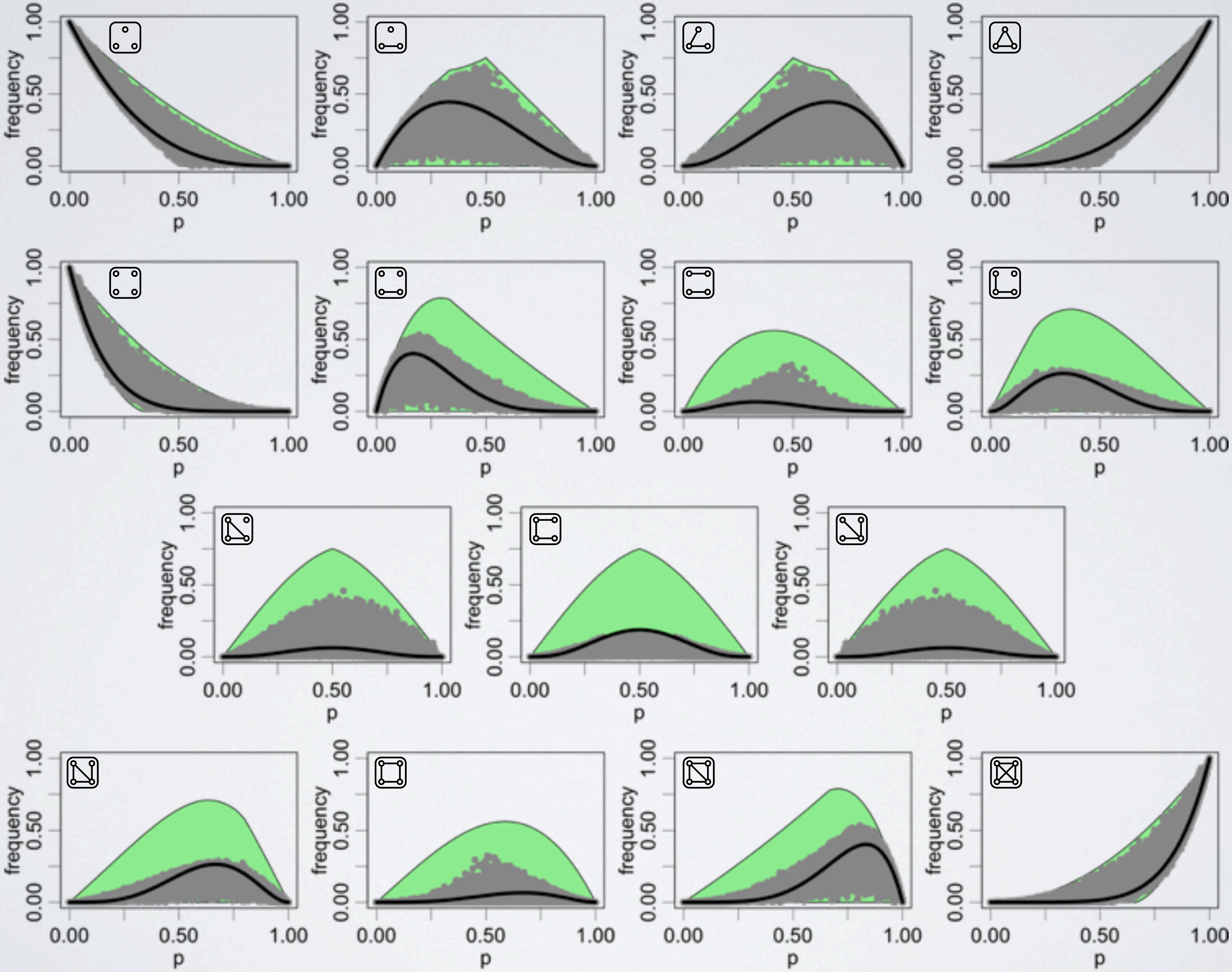
Subgraph frequencies

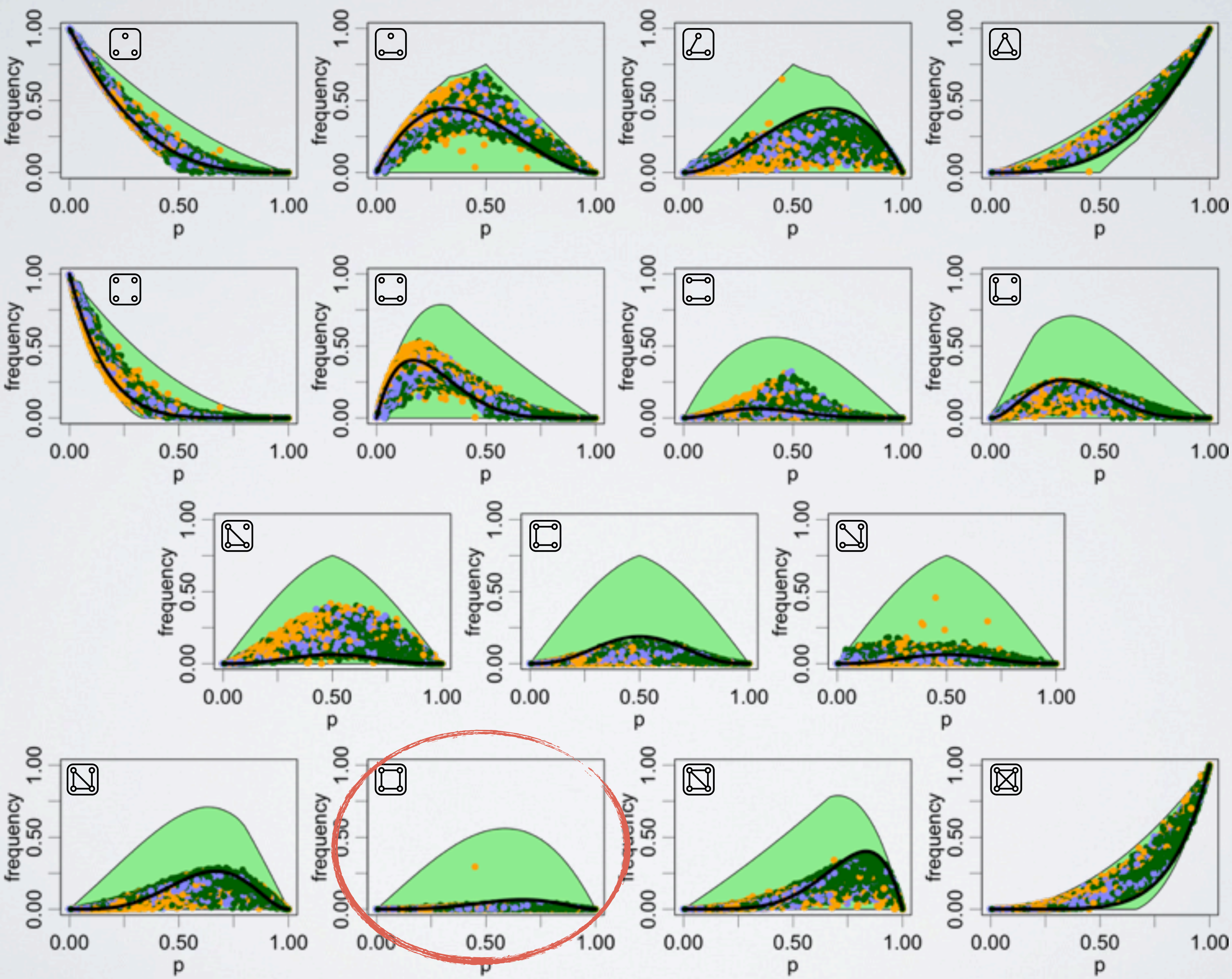


‘Crowd-sourced’ inner bounds



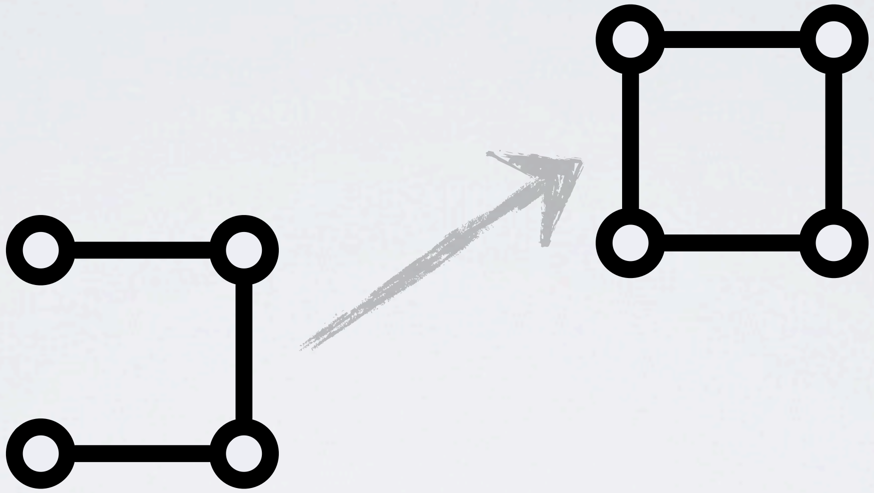
Consider all social graphs and the complements of all graphs, anti-social graphs (which are also graphs!)

What graphs are missing?



Triadic Closure and Squares

- Square unlikely to form:



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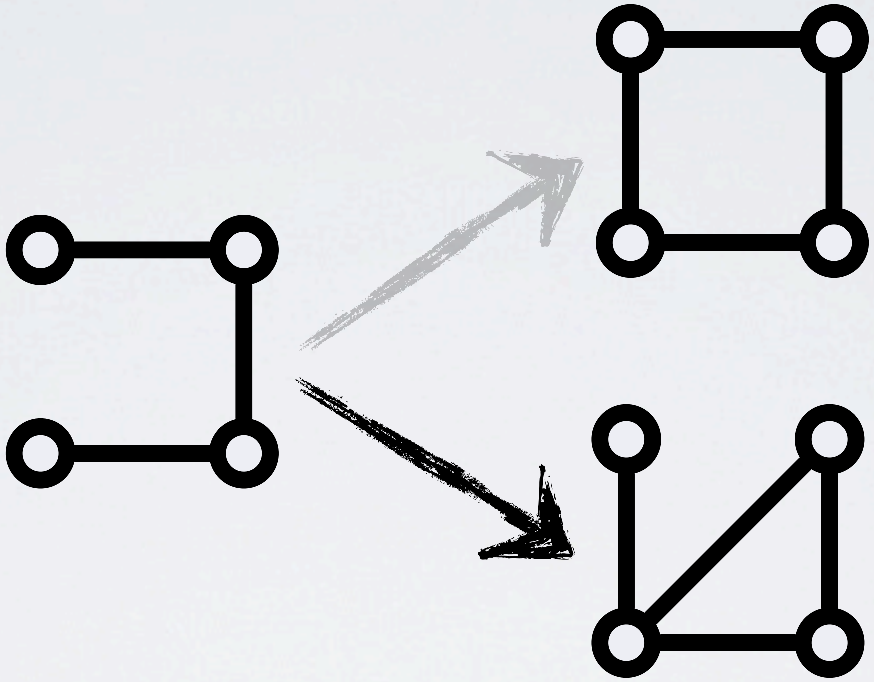

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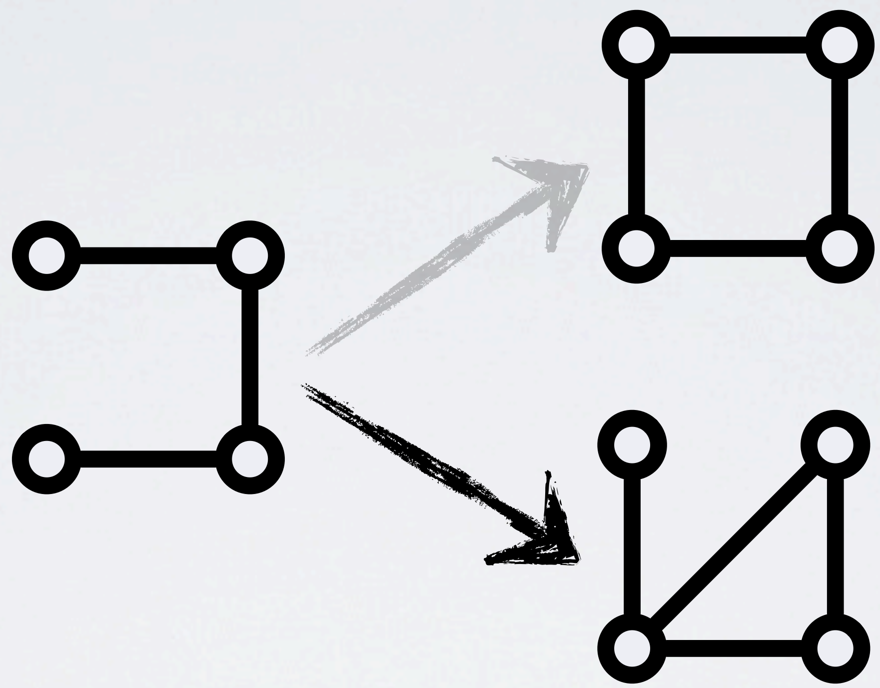
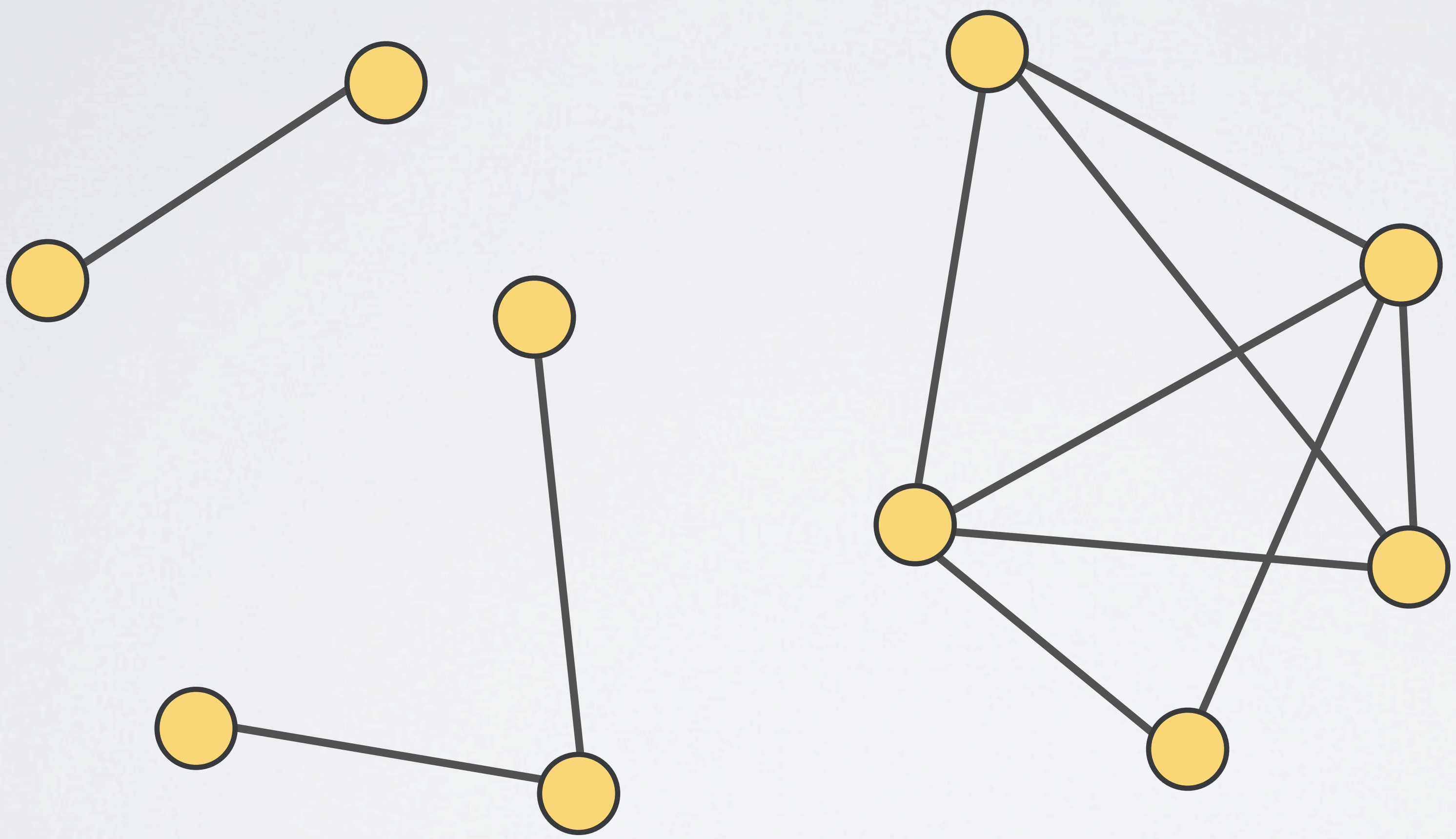


- Square has very short ‘**half-life**’:

Continuous Time Markov Chain Model



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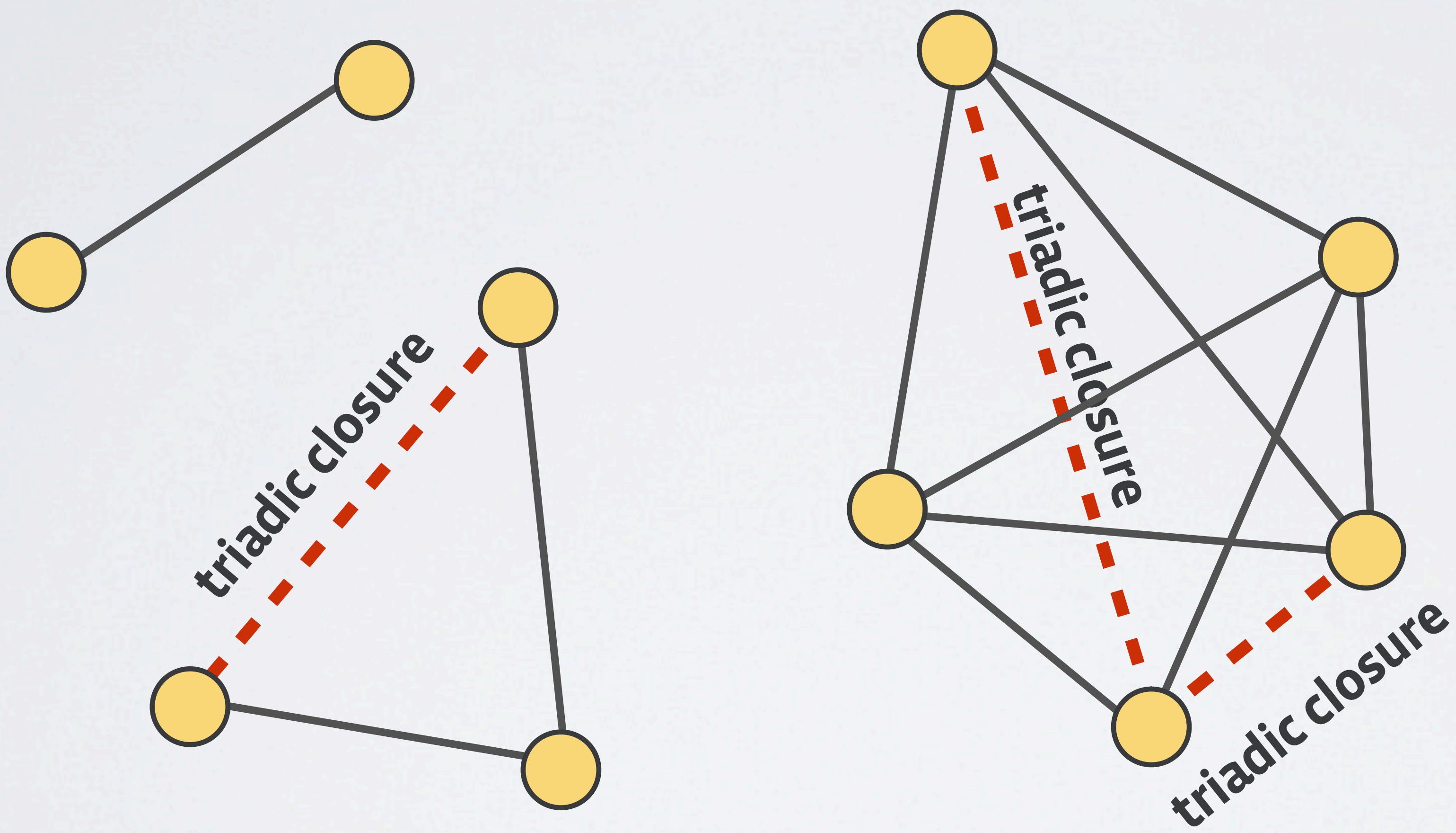

Edge Formation Random Walk (EFRW)

- **Continuous-time Markov chain**
- Transitions between unlabeled, undirected graphs based in edge formation.

- Independent **Poisson processes** for all node pairs:
  - Arbitrary formation: rate $\gamma > 0$ 
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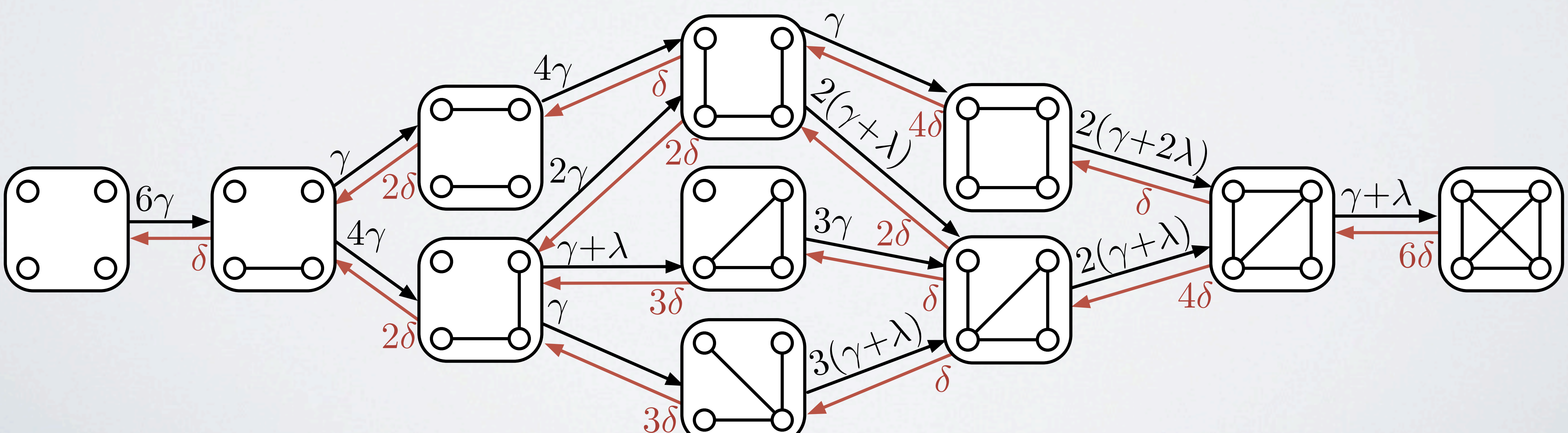

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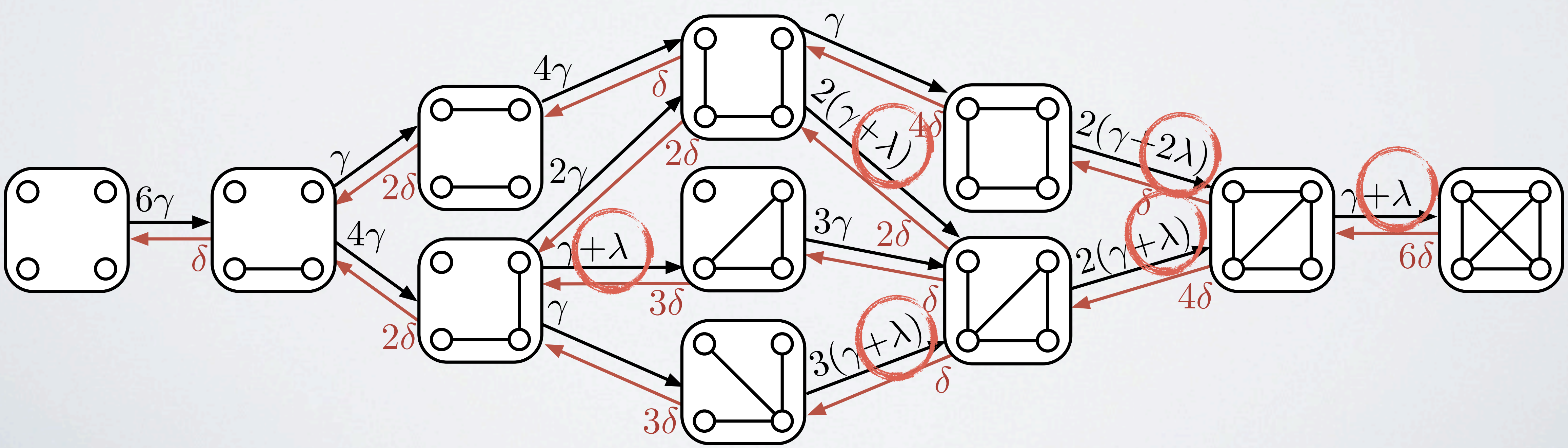
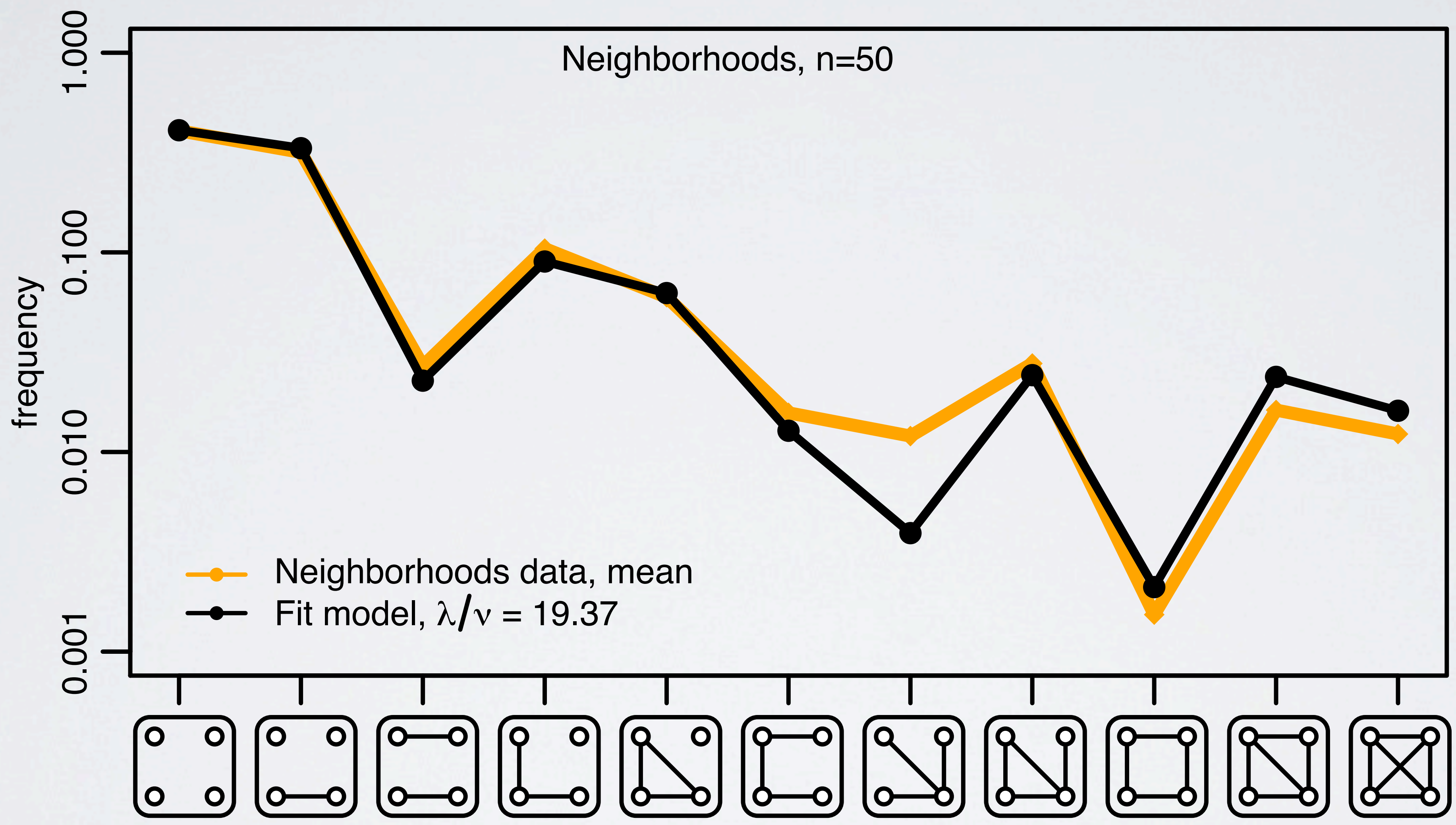


Fitting λ to subgraph data

- How well can we fit λ?

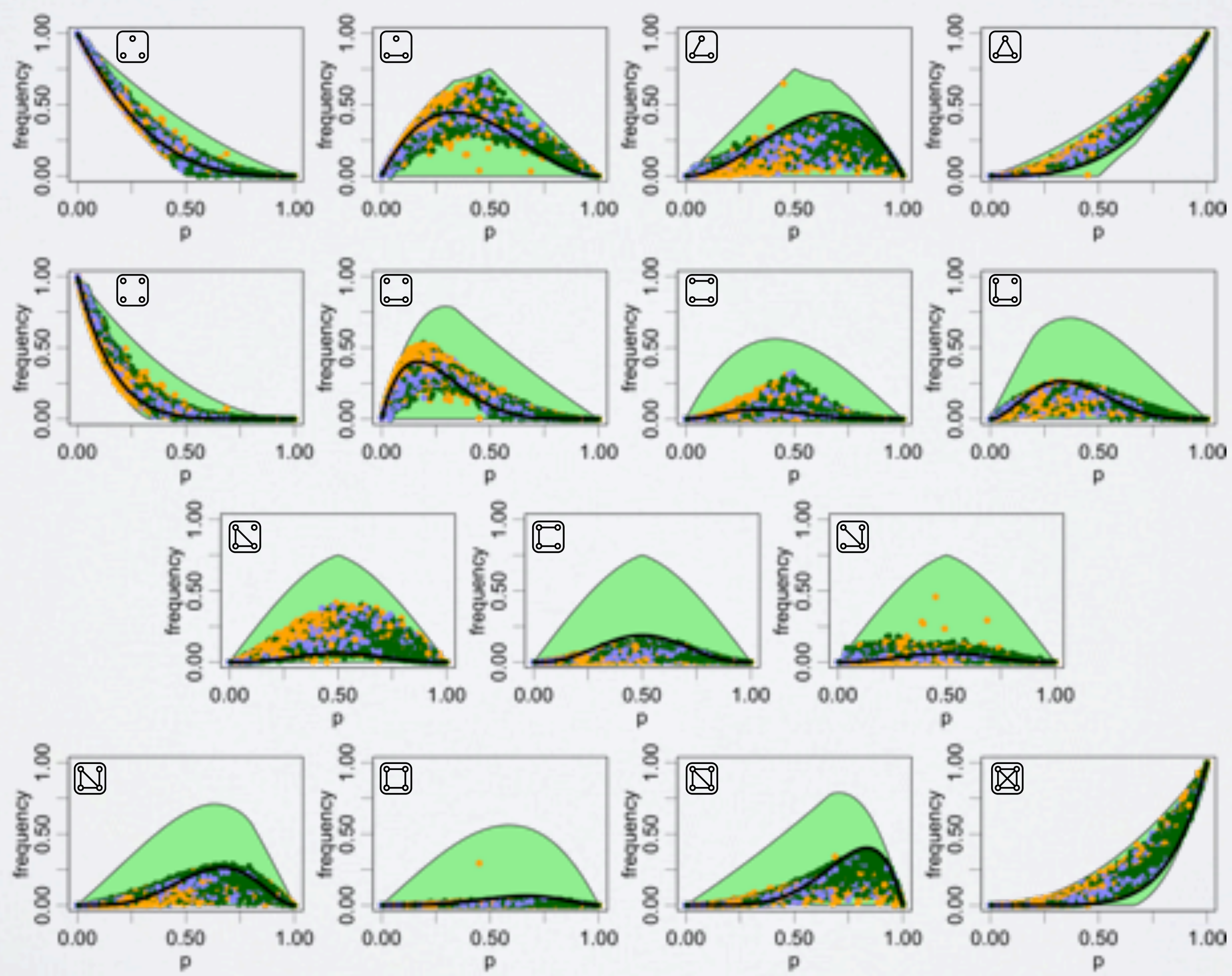


(log-scale y-axis)

- Subgraph frequencies are modeled very well by triadic closure.

Extremal graph theory

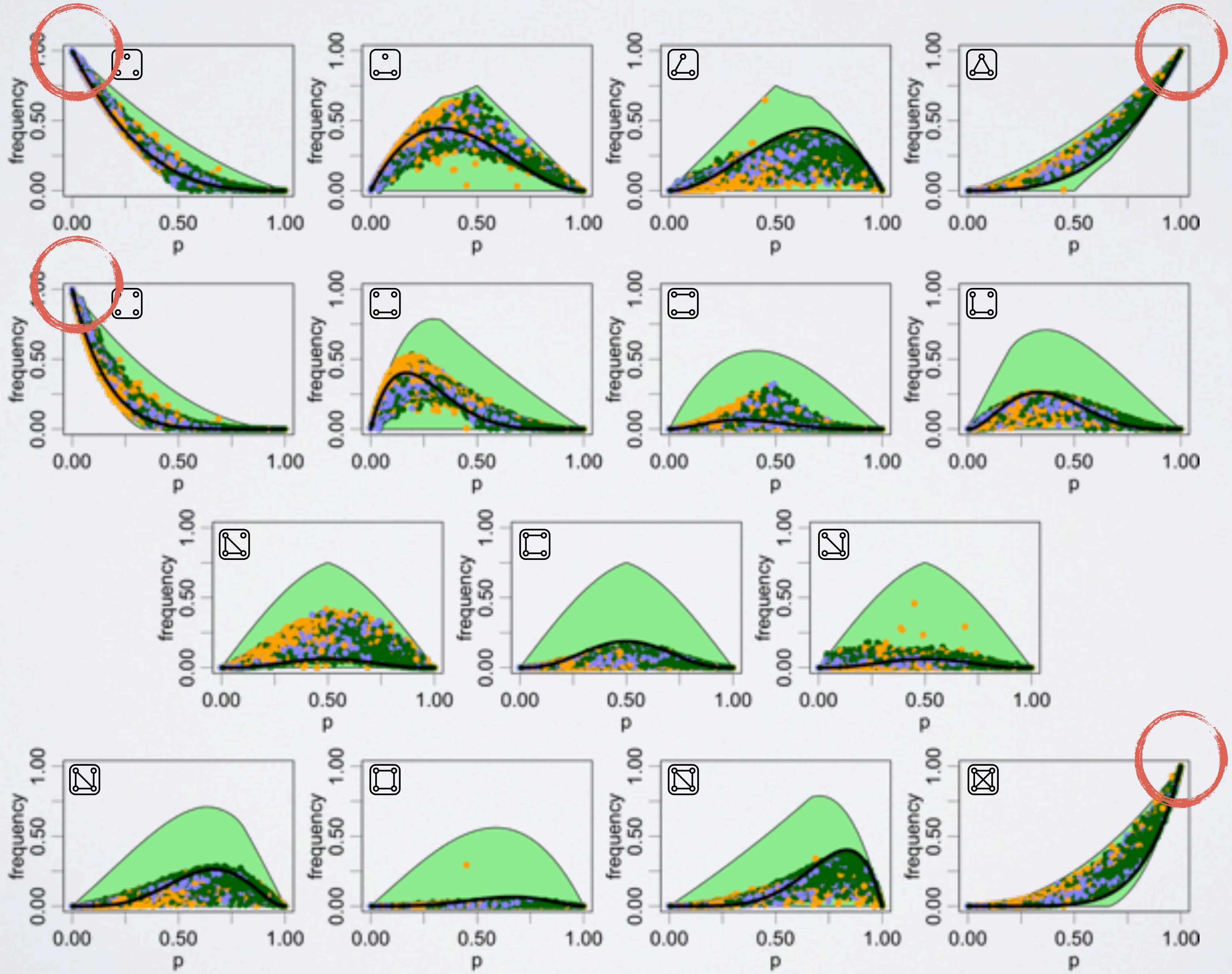
- Subgraph frequencies **s(F,G)** closely related to homomorphism density **t(F,G)**.
  [Borgs et al. 2006, Lovasz 2009]
- Frequency of cliques, lower bounds: **Moon-Moser** 1962, **Razborov** 2008
- Frequency of cliques, upper bounds: **Kruskal-Katona Theorem**
- Frequency of trees: **Sidorenko Conjecture** ('Theorem for trees')
- Also linear relationships across sizes.
- => Linear Program!



Extremal graph theory

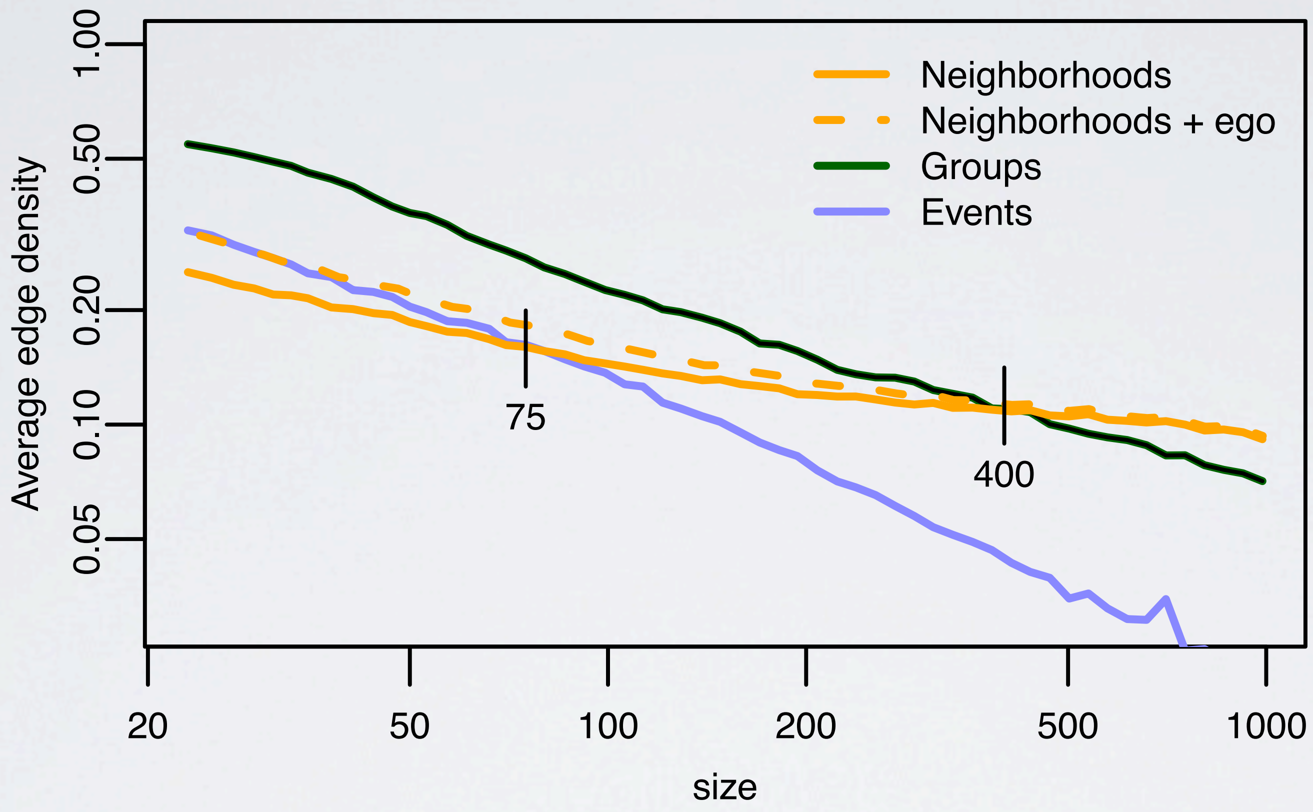
- A proposition for all subgraphs:

Proposition. For every k, there exist constants ϵ and n_0 such that the following holds. If F is a k-node subgraph that is not a clique and not empty, and G is any graph on $n \geq n_0$ nodes, then $s(F, G) < 1 - \epsilon$.



Audience graph classification

- How do different audience graphs differ?



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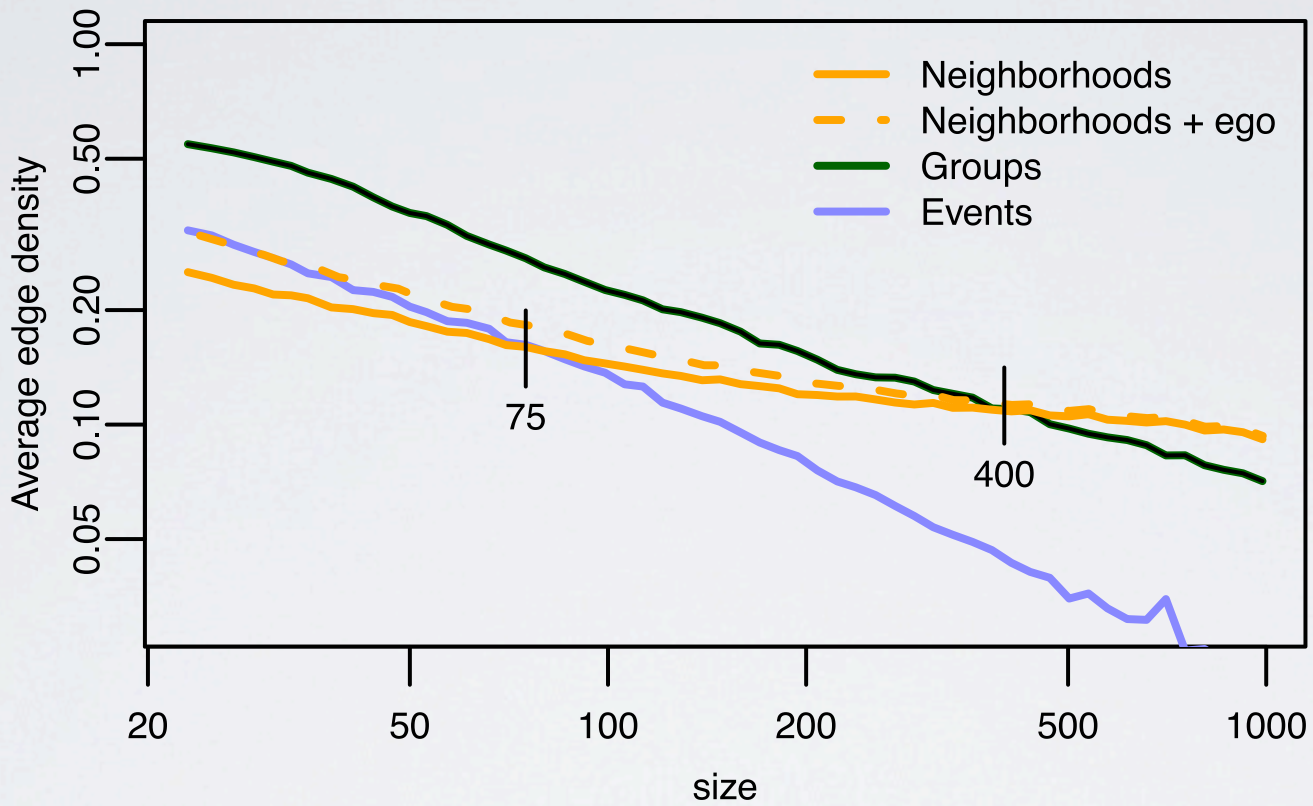

- Classification challenges

A) 75-node neigh. vs. 75-node events
B) 400-node neigh. vs. 400-node groups

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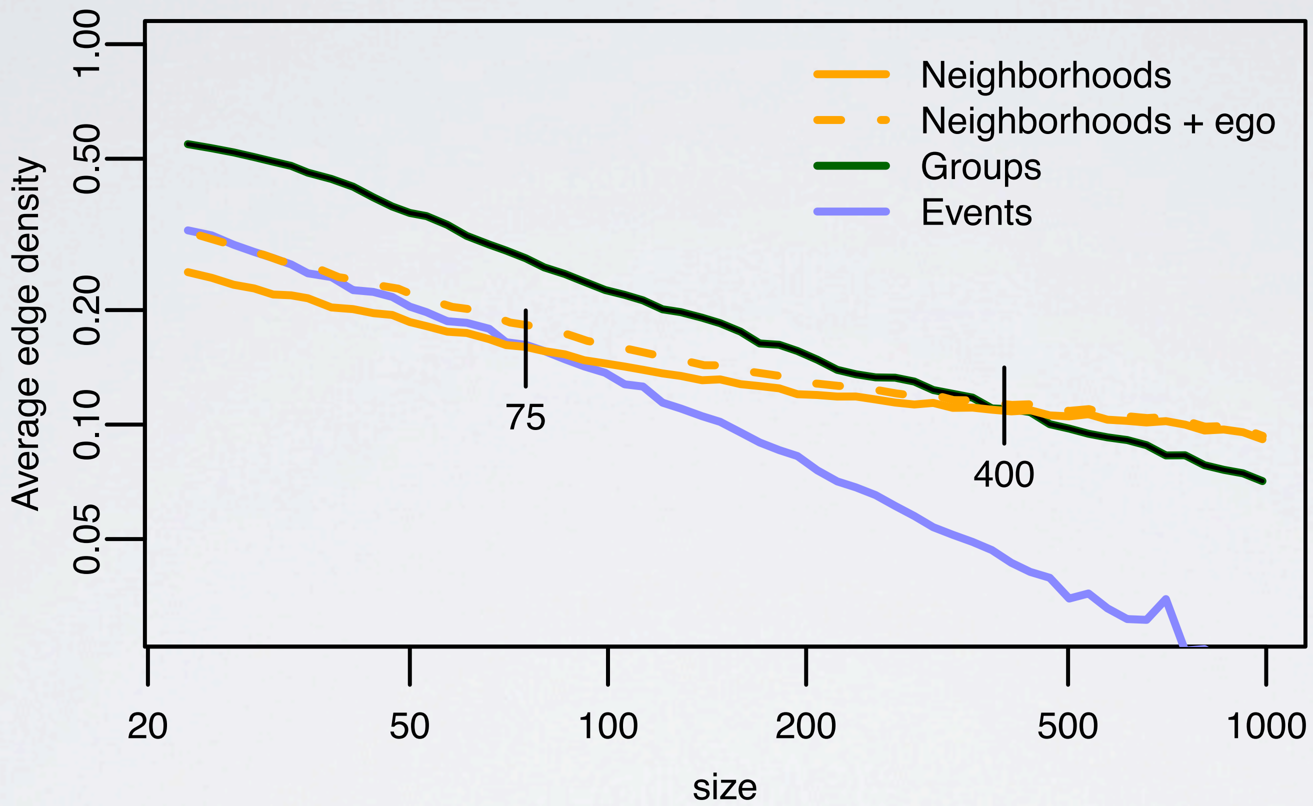

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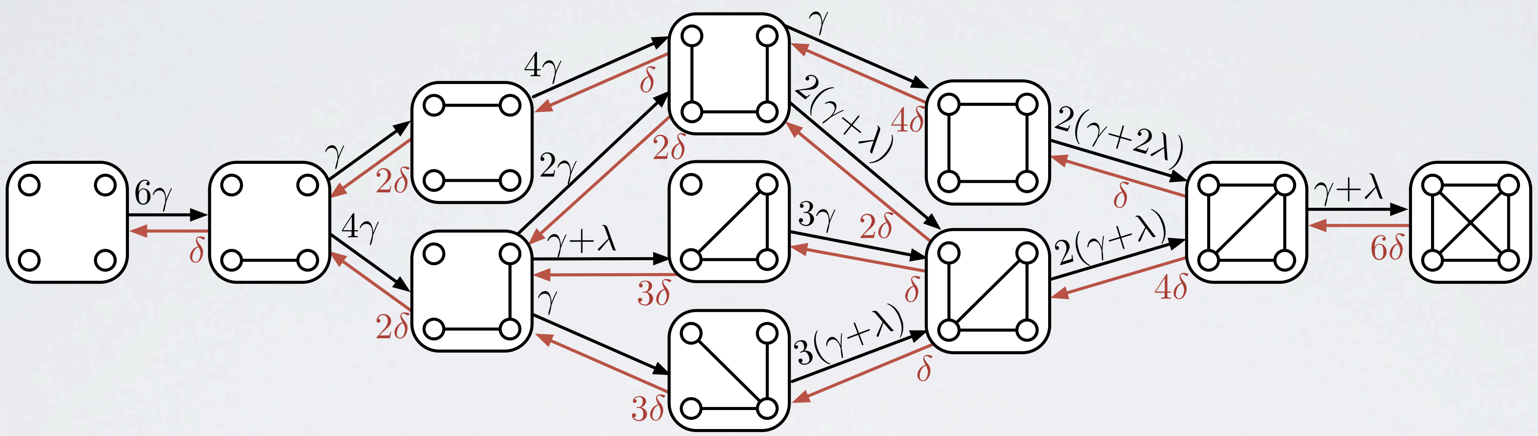
- Features:

Quad frequencies :	76% / 76%	accuracy
Global features:	69% / 76%	accuracy
Quad frequencies + Global features:	81% / 82%	accuracy

Conclusions

- Subgraph frequencies usefully characterize social graphs, have extremal limits!
- Edge Formation Random Walk model of dense social graphs:



- Homomorphism density bounds yield subgraph density bounds: